Interactions of disparate scales in drift-wave turbulence

John A. Krommes* and Chang-Bae Kim†
Princeton University, P.O. Box 451, Princeton, New Jersey 08543-0451
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Renormalized statistical theory is used to calculate the interactions between short scales (wave vector $\boldsymbol{k}$) and long scales (wave vector $\boldsymbol{q} \ll \boldsymbol{k}$) in the Hasegawa-Mima model of drift-wave turbulence (generalized to include proper nonadiabatic response for $k_{\parallel}=0$ fluctuations). The calculations include the zonal-flow growth rate as a special case, but also describe long-wavelength fluctuations with $\boldsymbol{q}$ oriented at an arbitrary angle to the background gradient. The results are fully renormalized. They are subtly different from those of previous authors, in both mathematical form and physical interpretation. A term arising in previous treatments that is related to the propagation of short-scale wave packets is shown to be a higher-order effect that must consistently be neglected to lowest order in a systematic expansion in q/k. Rigorous functional methods are used to show that the long-wavelength growth rate $\gamma_{\boldsymbol{q}}$ is related to second-order functional variations of the short-wavelength energy and to derive a heuristic algorithm. The principal results are recovered from simple estimates involving the first-order wave-number distension rate $\widetilde{\gamma}_{\boldsymbol{k}}^{(1)} \doteq \boldsymbol{k} \cdot \nabla \widetilde{\Omega}_{\boldsymbol{k}}/k^2$, where $\widetilde{\Omega}_{\boldsymbol{k}}$ is a nonlinear random advection frequency. Fokker-Planck analysis involving $\widetilde{\gamma}_{\boldsymbol{k}}^{(1)}$ is used to heuristically recover the evolution equation for the small scales, and a random-walk flux argument that relates $\widetilde{\gamma}_{\boldsymbol{k}}^{(1)}$ to an effective autocorrelation time is used to give an independent calculation of $\gamma_{\boldsymbol{q}}$. Both the rigorous and heuristic derivations demonstrate that the results do not depend on, and cannot be derived from, properties of linear normal modes; they are intrinsically nonlinear. The importance of random-Galilean-invariant renormalization is stressed.

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I. INTRODUCTION

It is widely believed that random zonal flows play an important role in determining the steady state and transport levels of drift-wave and Rossby-wave turbulence. In a slab geometry with profile gradients in the x direction and magnetic field in the z direction, zonal flows are defined to be the y-directed $\boldsymbol{E}\times\boldsymbol{B}$ velocities that result from electrostatic potentials with wave numbers $(q_x, q_y, q_z) = (q_x, 0, 0)$. Such flows can be driven by nonlinear mode coupling, e.g.,

$$(q_x,0,0) = (k_x, k_y, k_z) + (-k_x + q_x, -k_y, -k_z). \tag{1}$$

In the present paper we calculate the short-wavelength-driven contribution to the long-wavelength nonlinear growth rate $\gamma_{\boldsymbol{q}}$ of zonal flows and other long-wavelength fluctuations by proceeding systematically from well-established theories of statistical dynamics applied to a generalization of the Hasegawa-Mima (HM) paradigm of nonlinear drift waves. (It is well known that the HM system is homologous to simple models of Rossby waves [1], so our results are applicable to certain problems in geostrophic physics as well.) When specialized to zonal flows, some of our formulas are quite similar in form to ones proposed heuristically in Ref. [2]. They are not identical, however, and the differences are both conceptually interesting and quantitatively significant. Without an underlying systematic derivation, it would be difficult on purely heuristic or dimensional grounds to argue for one form over the other. Therefore, we present our calculations in considerable detail and from several different routes. We also derive from first principles several heuristic algorithms for $\gamma_{\boldsymbol{q}}$ whose interpretations differ in fundamental physical ways from earlier suggestions in the literature.

Let $\boldsymbol{q}$ be a characteristic wave vector of the long scales (such as zonal flows); similarly, let $\boldsymbol{k}$ be a typical wave vector of the short scales (frequently associated with drift-wave turbulence). In general, the scales of zonal flows may be comparable to drift-wave scales ($q_x \sim k_x$), and such fluctuations have been observed in computer simulations [3,4]. However, the assumption of disparate scales ($q \ll k$) is a useful device that enables analytical progress. (Some workers have also attempted to argue that the long wavelengths may be more effective in regulating the saturation of the short scales in some situations, but this assertion is controversial and does not motivate the present work.) Then the key ordering parameter is $\epsilon \doteq q/k \ll 1$ (where $\doteq$ denotes definition). It is important to note that ϵ does not depend on the dynamical properties of the fluctuations. Of course, those properties must enter any calculation, since they are contained in the primitive amplitude equations whose statistics are studied, but it is the small-q assumption that enables one to simplify the general formulas. Therefore, one need not focus on zonal flows *per se*. The formulas we obtain are also applicable to the generation of random streamers [$\boldsymbol{q} = (0, q_y, 0)$] as well as to other long-wavelength fluctuations with arbitrary (small) $\boldsymbol{q}$.

The interactions of disparate scales have a long history in statistical turbulence theory. In fundamental work, Kraichnan [5] gave an exact definition of eddy viscosity for homogeneous, isotropic Navier-Stokes (NS) turbulence, and he discussed its properties for two- and three-dimensional tur-

*Email address: krommes@princeton.edu
†Permanent address: Soongsil University, 1-1 Sangdo5-dong Dongjak-gu, Seoul, Korea. Email address: cbkim@plasma.soongsil.ac.kr

bulence. His eddy viscosity $\nu(q|k_{\mathrm{m}})$ describes, in a statistical sense, the effective turbulent damping of resolved modes with wave number q due to interactions with unresolved modes having wave numbers $k \geqslant k_{\mathrm{m}}$. Given a statistical closure such as the test-field model (TFM) [6], an exact formula can be given for ν. Kraichnan found an appealing approximate result [our Eq. (64) below] by expanding in powers of $q/k_{\mathrm{m}} \ll 1$. His results for two-dimensional (2D) isotropic turbulence are particularly relevant in the context of drift waves. Indeed, some of the algebra described in the present paper is essentially merely an extension of Kraichnan's calculations to anisotropic quasi-2D turbulence. However, we also give alternative calculational procedures and physical interpretations that should be of interest in the contexts of both NS and plasma turbulence, and we attempt to clarify a number of confusions in the literature.

For 2D isotropic NS turbulence, Kraichnan showed that the eddy viscosity is negative under quite broad circumstances; we show that this same conclusion holds for anisotropic HM dynamics as well. Recently Chechkin *et al.* [7] attempted to calculate eddy viscosity for Rossby- and drift-wave turbulence, generalizing earlier work of Montgomery and Hatori [8] on 2D NS flows. The results of those authors disagree with that of Kraichnan; they found a positive eddy viscosity in the 2D NS limit. Our present work confirms Kraichnan's result. We explain the source of the discrepancy in the Appendix.

We shall consider the statistical interactions between long and short scales for the specific model of HM dynamics, with an appropriately modified Poisson equation for $k_\parallel = 0$ Fourier components. (Here $k_\parallel$ is the component of $\boldsymbol{k}$ in the direction of the magnetic field $\boldsymbol{B}$.) By direct expansion in ϵ of well-known general formulas for Markovian statistical closure, we obtain formulas for the nonlinear growth rate $\gamma_{\boldsymbol{q}}$ of the long-wavelength modes, the nonlinear noise acting on those modes, and the corresponding energy-conserving terms for the evolution of the short-wavelength fluctuations. Specific results for zonal flows can be trivially extracted as a special case. Our formulas are fully renormalized, apply uniformly to regimes of both weak and strong turbulence, and reduce properly to Kraichnan's result for 2D isotropic NS turbulence. We provide two versions of the direct calculations: one appropriate for isotropic statistics; the other valid for arbitrary anisotropy. As a nontrivial cross-check, we verify that the anisotropic results reduce correctly to the isotropic ones.

Although the direct reduction of the general Markovian formulas is relatively concise and the final forms of the results are suggestive, the algebraic details are not particularly physically illuminating. Accordingly, we derive from first principles several heuristic algorithms and demonstrate that they recover the correct $\gamma_{\boldsymbol{q}}$ to lowest order in ϵ. Our work in this direction was strongly influenced by the earlier work of Diamond *et al.* [2], who for the special case of pure zonal flows made a well-motivated attempt to obtain a simple heuristic algorithm for $\gamma_{\boldsymbol{q}}$ by combining the use of a certain ''quasilinear'' Poynting theorem with an action-conservation principle. That algorithm (which was not derived from first principles) suggests that the results depend upon, and can be calculated from, the properties of the short-wavelength linear normal modes, i.e., the conventional electrostatic drift waves. In fact, however, the proper results have virtually nothing to do with short-wavelength linear theory; they are intrinsic properties of the nonlinear mode coupling. Thus, long-wavelength fluctuations have a nonzero nonlinear growth rate even in the complete absence of linear waves; cf. Kraichnan's results for 2D NS turbulence, the linear theory of which consists merely of weak viscous dissipation. In a beautiful and detailed physical explanation of his results, Kraichnan emphasized the importance of enstrophy conservation during the nonlinear interactions. The authors of Ref. [2] also focused on conservation properties (they did not cite Kraichnan's work). However, whereas they attempted to invoke a quasilinear wave-energy theorem, we will show that the proper derivation of $\gamma_{\boldsymbol{q}}$ from energetics requires use of a *nonlinear* energy-balance theorem. That is not a trivial modification of the quasilinear one to include small nonlinear corrections; it is a different version of the theorem that describes the intrinsically nonlinear interactions between disparate-scale fluctuations and has nothing to do with linear theory. Indeed, one finds that $\gamma_{\boldsymbol{q}}$ arises from *second-order* variations of the short-wavelength energy, consistent with nonlinear interactions in a random medium, and this observation motivates a heuristic algorithm that makes it clear that one must work with nonlinear fluctuations that are unrelated to linear normal modes. When applied to the 2D NS equation, our results are entirely compatible with Kraichnan's analysis, but are obtained from a somewhat more general point of view.

A key quantity that emerges from the analysis is the first-order wave-number distension rate $\tilde{\gamma}_{\boldsymbol{k}}^{(1)} \doteq \boldsymbol{k} \cdot \boldsymbol{\nabla} \tilde{\Omega}_{\boldsymbol{k}} / k^2$, where $\tilde{\Omega}_{\boldsymbol{k}}$ is an appropriate nonlinear advection frequency. This describes the evolution of the characteristic wave number of short-scale wave packets under a long-wavelength random modulation, $d \ln k^2/dt = -2\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$, which is responsible for the transfer of energy between short and long scales. We show that $\gamma_{\boldsymbol{q}}$ can be naturally written in terms of $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$, and also that a heuristic Fokker-Planck analysis recovers the systematically derived spectral evolution equation for the small scales. The significance of $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$ has not been previously recognized in the literature on zonal flows to our knowledge.

A. The Hasegawa-Mima model of nonlinear drift-wave dynamics

Hasegawa-Mima dynamics are a useful paradigm for the nonlinear interaction of drift waves. They emerge as a simple limit of the nonlinear gyrokinetic (GK) equation [9] in which the ion temperature and parallel motion are taken to vanish; the resulting GK continuity equation for ion gyrocenter density fluctuation n_i^G (normalized to background density N_i) is

$$\partial_t n_i^G + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} N_i + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} n_i^G = 0, \tag{2}$$

where in appropriately dimensionless units (cf. Ref. [10]) the $\boldsymbol{E} \times \boldsymbol{B}$ velocity is $\boldsymbol{V}_E \doteq \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} \varphi$, φ being the electrostatic potential. The system is closed by the GK Poisson equation [9]

$$\nabla_\perp^2 \varphi = -(n_i^G - n_e^G), \tag{3}$$

an expression of the quasineutrality condition $n_i = n_e$ (the latter quantities are the particle, not gyrocenter, densities).

The left-hand side of Eq. (3) describes the ion polarization charge density; the electron polarization is negligible, so $n_e^G \approx n_e$. For $k_\parallel \neq 0$ fluctuations the electrons are assumed to be adiabatic, $n_e = \varphi$, and one obtains the conventional $T_i \to 0$ limit of the gyrokinetic Poisson equation,

$$(1-\nabla_\perp^2)\varphi = n_i^G. \tag{4}$$

However, for $k_\parallel = 0$ fluctuations (a special case being zonal flows) electron parallel response is inhibited and a standard, justifiable approximation [11,12] is to drop the 1 from Eq. (4). The resulting dynamical system, a generalization of the familiar equation of Hasegawa and Mima [13], can be written as

$$\partial_t \varphi + (\hat{\alpha}-\nabla_\perp^2)^{-1} V_* \partial_y \varphi + (\hat{\alpha}-\nabla_\perp^2)^{-1} \boldsymbol{V}_E \cdot \boldsymbol{\nabla}[(\hat{\alpha}-\nabla_\perp^2)\varphi] = 0 \tag{5}$$

(where $\hat{\alpha}$ vanishes for $k_\parallel = 0$ and is the identity operator otherwise, and where the diamagnetic velocity is defined by $V_* \doteq L_n^{-1} \doteq -\partial_x \ln N$) or, upon Fourier analysis in space (assuming constant V_*), in the standard form

$$\partial_t \varphi_{\boldsymbol{k}} + i\Omega_{\boldsymbol{k}}^{\text{lin}} \varphi_{\boldsymbol{k}} = \frac{1}{2} \sum_{\Delta} M_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \varphi_{\boldsymbol{p}}^* \varphi_{\boldsymbol{q}}^*, \tag{6}$$

where

$$\Omega_{\boldsymbol{k}}^{\text{lin}} \doteq \frac{\omega_*(\boldsymbol{k})}{\alpha_{\boldsymbol{k}} + k^2}, \tag{7a}$$

$$M_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \doteq \frac{(\hat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{q})[(\alpha_{\boldsymbol{q}}+q^2)-(\alpha_{\boldsymbol{p}}+p^2)]}{\alpha_{\boldsymbol{k}}+k^2}, \tag{7b}$$

$$\alpha_{\boldsymbol{k}} \doteq \begin{cases} 1 & (k_\parallel \neq 0) \\ 0 & (k_\parallel = 0), \end{cases} \tag{7c}$$

$\omega_*(\boldsymbol{k}) \doteq k_y V_*$, $k \equiv |\boldsymbol{k}_\perp|$, and $\Sigma_\Delta \equiv \Sigma_{\Delta(\boldsymbol{k};\boldsymbol{p},\boldsymbol{q})}$ denotes the sum over all $\boldsymbol{p}$ and $\boldsymbol{q}$ such that $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}$. For the later algebra, it will be useful to adopt the shorthand notation $\overline{k^2} \doteq \alpha_{\boldsymbol{k}} + k^2$; thus

$$M_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} = (\hat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{q})(\overline{q^2}-\overline{p^2})/\overline{k^2}. \tag{8}$$

This defines what we call *generalized HM dynamics*. In the original HM approximation, all α's were set to 1, giving rise to the HM mode-coupling coefficient

$$M_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}^{\text{HM}} \doteq \frac{(\hat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{q})(q^2-p^2)}{1+k^2}. \tag{9}$$

This defines what we call *pure HM dynamics*. For 2D NS turbulence, all α's vanish.

For generalized HM dynamics, we define the two weight factors $\sigma_{\boldsymbol{k}}^{(E)}$ and $\sigma_{\boldsymbol{k}}^{(Z)}$ according to

$$\begin{pmatrix} \sigma_{\boldsymbol{k}}^{(E)} \\ \sigma_{\boldsymbol{k}}^{(Z)} \end{pmatrix} \doteq \frac{1}{2} \begin{pmatrix} 1 \\ \overline{k^2} \end{pmatrix} \overline{k^2}. \tag{10}$$

One then has the detailed conservation properties

$$\sigma_{\boldsymbol{k}}^{(Q)} M_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} + \text{c.p.} = 0 \quad (Q = E \text{ or } Z), \tag{11}$$

where c.p. denotes the cyclic permutations $\boldsymbol{k}\to\boldsymbol{p}\to\boldsymbol{q}$. These guarantee that the nonlinear term of Eq. (6) conserves both the primitive energy $\tilde{E}$ and the generalized enstrophy [14] $\tilde{Z}$, where

$$\begin{pmatrix} \tilde{E} \\ \tilde{Z} \end{pmatrix} = \sum_{\boldsymbol{k}} \begin{pmatrix} \tilde{E}_{\boldsymbol{k}} \\ \tilde{Z}_{\boldsymbol{k}} \end{pmatrix}, \tag{12a}$$

$$\begin{pmatrix} \tilde{E}_{\boldsymbol{k}} \\ \tilde{Z}_{\boldsymbol{k}} \end{pmatrix} \doteq \begin{pmatrix} \sigma_{\boldsymbol{k}}^{(E)} \\ \sigma_{\boldsymbol{k}}^{(Z)} \end{pmatrix} |\delta\varphi_{\boldsymbol{k}}|^2. \tag{12b}$$

Here the tilde denotes a random variable (or a property of a particular realization of the turbulence). For later discussions of pure HM dynamics, we shall also define the potential enstrophy $\tilde{W}$ such that

$$\tilde{W}_{\boldsymbol{k}} \doteq \sigma_{\boldsymbol{k}}^{(W)} |\delta\varphi_{\boldsymbol{k}}|^2, \quad \sigma_{\boldsymbol{k}}^{(W)} \doteq \tfrac{1}{2} k^2 \overline{k^2} = k^2 \sigma_{\boldsymbol{k}}^{(E)}. \tag{13}$$

For $k_\parallel \neq 0$ fluctuations, one has $\tilde{Z} = \tilde{E} + \tilde{W}$; in pure HM dynamics, $\tilde{W}$ is conserved. Also note that $\overline{k^2}$ is the ratio between generalized enstrophy and energy: $\tilde{Z}_{\boldsymbol{k}} = \overline{k^2} \tilde{E}_{\boldsymbol{k}}$.

Another way of describing the physical content of Eq. (5) is to decompose φ into a $k_\parallel = 0$ part $\overline{\varphi}$ and a $k_\parallel \neq 0$ part $\breve{\varphi}$: $\varphi = \overline{\varphi} + \breve{\varphi}$. Thus, $\hat{\alpha}$ is a projection operator onto the $k_\parallel \neq 0$ subspace: $\breve{\varphi} = \hat{\alpha}\varphi$. Then

$$\partial_t \varphi + (\hat{\alpha}-\nabla_\perp^2)^{-1} V_* \partial_y \varphi + (\hat{\alpha}-\nabla_\perp^2)^{-1} \boldsymbol{V}_E \cdot \boldsymbol{\nabla}(-\nabla_\perp^2 \varphi) + \underline{(\hat{\alpha}-\nabla_\perp^2)^{-1} \overline{\boldsymbol{V}}_E \cdot \boldsymbol{\nabla} \breve{\varphi}} = 0. \tag{14}$$

For pure HM dynamics, $\hat{\alpha} = 1$ and the underlined correction term disappears. The importance of that term has been recognized in Refs. [2] and [15]; it involves the so-called $\boldsymbol{E}\times\boldsymbol{B}$ nonlinearity as opposed to the so-called polarization-drift nonlinearity of HM. Its significance is that it destroys the pure HM $\tilde{W}$ invariant, replacing it by $\tilde{Z}$.

We have specifically omitted linear dissipation and/or growth in Eq. (6) so that one can focus on intrinsically nonlinear effects and avoid being confused by possible linear phase shifts, which turn out to be largely irrelevant [they affect only the detailed form of the mode-mode interaction time θ, Eq. (17)]. Actually we could omit the diamagnetic term proportional to V_* as well. We retain it because it defines the conventional HM model, but it will also disappear from the final answer. Frequently Eq. (4) is generalized to include nonadiabatic electron response $[n_e = (1-i\hat{\delta})\varphi]$, which introduces a linear growth rate due to inverse Landau damping, and Eq. (6) is supplemented by an artificially inserted dissipation term; such models are variants of the so-called Terry-Horton (TH) equation [16]. In the absence of linear growth and damping, one does not obtain realistic states of forced, dissipative turbulence with the well-known dual cascades of energy and enstrophy [17]; the basic dissipation-free HM equation will relax to thermal equilibrium. However, with one possible exception to be noted in the next paragraph, that distinction will not affect the form of our final answers either. Thus we study a minimal, intrinsically nonlinear model.

Some general background on the physics of the HM and related equations is given in Ref. [18]. As is well known, the most important difference between the TH equation and the HM equation is that in the latter the mode-coupling coefficients are real whereas in the former they are complex because of the dissipative nonadiabatic response. The consequence is that separate $\tilde{E}$ and $\tilde{W}$ conservation is lost; only a single hybrid invariant $\tilde{Z}_\delta$ [approximately equal to the $\tilde{Z}$ defined by Eqs. (12)] survives. This affects certain long-time properties of the turbulence, and the asymptotic behavior of the TH system in the limit of small $\hat{\delta}$ is peculiar. However, we do not believe that our results are affected by ignoring a small dissipative $\hat{\delta}$. (We retain strongly nonadiabatic response for the $k_\parallel=0$ modes.) In any event, the generalized HM model enables us to demonstrate the techniques for calculating the interactions of disparate scales with a minimum of complications; it is a definite and interesting dynamical system in its own right.

B. Markovian statistical closures

Now consider the statistical description of Eq. (6). (Some background on statistical closures can be found in Refs. [19–21], each of which contains many further references.) Since V_* is assumed to be a constant independent of space, one may assume periodic boundary conditions and appropriate initial statistics such that there is no mean field: $\langle\varphi\rangle=0$. The statistical description of Eq. (6) then reduces to a theory of the fluctuation $\delta\varphi\doteq\varphi-\langle\varphi\rangle$ with spatially homogeneous statistics. (It is necessary to emphasize this point because later we will consider particular kinds of perturbations away from the homogeneous state.) With that assumption, the general form of a Markovian spectral balance equation [20] for a single scalar field φ evolving under quadratic nonlinearity is

$$\partial_t C_{\boldsymbol{k}}+2\,\mathrm{Re}\,\eta_{\boldsymbol{k}}C_{\boldsymbol{k}}=2F^{\mathrm{nl}}_{\boldsymbol{k}}, \tag{15}$$

where $C_{\boldsymbol{k}}\doteq\langle|\delta\varphi_{\boldsymbol{k}}|^2\rangle$, $\eta_{\boldsymbol{k}}\doteq i\omega^{\mathrm{lin}}_{\boldsymbol{k}}+\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ is a coherent (generally complex) damping [$\omega^{\mathrm{lin}}_{\boldsymbol{k}}\doteq\Omega^{\mathrm{lin}}_{\boldsymbol{k}}+i\gamma^{\mathrm{lin}}_{\boldsymbol{k}}$ contains the linear frequency and growth rate, the latter being absent from Eq. (6)], and $F^{\mathrm{nl}}_{\boldsymbol{k}}$ is the covariance of an internally produced incoherent noise. Realizable Langevin representations underlying Eq. (15) were discussed in Refs. [22], [23], and [20]; see also the related discussion in Ref. [24]. In the eddy-damped quasinormal Markovian closure [25] and the steady-state limit of the realizable Markovian closure [20], one obtains

$$\eta^{\mathrm{nl}}_{\boldsymbol{k}}\doteq-\sum_{\Delta}M_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}M^*_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}}\theta^*_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}C_{\boldsymbol{q}}, \tag{16a}$$

$$F^{\mathrm{nl}}_{\boldsymbol{k}}\doteq\frac{1}{2}\sum_{\Delta}|M_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}|^2\,\mathrm{Re}\,\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}C_{\boldsymbol{p}}C_{\boldsymbol{q}}, \tag{16b}$$

where $\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$ is the triad interaction time (assumed here to be a symmetrical function of its arguments) described in the next paragraph. It is readily shown that the forms (16) preserve the same quadratic invariants $E\doteq\langle\tilde{E}\rangle$ and $Z\doteq\langle\tilde{Z}\rangle$ as do the primitive amplitude equations, by virtue of the detailed conservation properties (11).

In steady state, one has

$$\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}=(\eta^S_{\boldsymbol{k}}+\eta^S_{\boldsymbol{p}}+\eta^S_{\boldsymbol{q}})^{-1}, \tag{17}$$

where $\eta^S_{\boldsymbol{k}}$ is a modified version of $\eta_{\boldsymbol{k}}$, as discussed in the next paragraph. (S stands for solenoidal; for the closures cited so far, $\eta^S_{\boldsymbol{k}}=\eta_{\boldsymbol{k}}$.) The general form of Eq. (17) is intuitively plausible in view of the primitive triadic couplings stemming from Fourier analysis of the quadratic nonlinearity. More formally, θ is defined in terms of a mean response function $R^S_{\boldsymbol{k}}(t;t')$ that describes the averaged dynamical response of mode $\boldsymbol{k}$ at time t due to an infinitesimal perturbation at time t':

$$\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}(t)=\int_{-\infty}^{t}dt'\,R^S_{\boldsymbol{k}}(t;t')R^S_{\boldsymbol{p}}(t;t')R^S_{\boldsymbol{q}}(t;t'). \tag{18}$$

Since

$$\partial_t R^S_{\boldsymbol{k}}(t;t')+\eta^S_{\boldsymbol{k}}R^S_{\boldsymbol{k}}=\delta(t-t'), \tag{19}$$

one is led in steady state directly to Eq. (17). It is worth noting that the Markovian theories build in the fluctuation-dissipation ansatz, which in steady state reads [26,27]

$$C_{\boldsymbol{k}}(\tau)=R_{\boldsymbol{k}}(\tau)C_{\boldsymbol{k}}(0)\quad(\tau>0). \tag{20}$$

Some subtle issues relating to the form of that ansatz for transient evolution were discussed in Ref. [20]; however, they will not play a role in the present calculations, which can be taken to be in steady state.

In the previous equations, the solenoidal qualification arises because of an important physics issue relating to the derivation of Eq. (17). The cited closures are close relatives of Kraichnan's direct-interaction approximation (DIA) [28], so they inherit the problems of that approximation with random Galilean invariance [29]. This is not an issue when the interacting scales are of comparable size; however, it is of great importance when interactions between disparate scales are considered, as in the present work. As Kraichnan has emphasized in the derivation of his test-field model [6], the random advection of small-scale eddies by large-scale ones is dominated by the mean-square *shear* in the large scales, not the large-scale energy. A consequence is that in the derivations of the effective interaction time for the spectral balance equation the η^{nl}'s must be calculated with mode-coupling coefficients modified at small wave numbers. Kraichnan accomplished that heuristically by tying the decorrelation effects to the behavior of the solenoidal part of a test field advected by the turbulence. Bowman and Krommes [30] discussed the issue in a context similar to the present one in the course of deriving a test-field model that remained realizable in the presence of linear waves.

In the present paper, we are concerned with the structural forms and physical interpretations of $\gamma_{\boldsymbol{q}}$ and related quantities, not with quantitative calculations. Accordingly, we write all of the results in terms of a given $\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$ whose properties are qualitatively well understood [5], so we do not need to face the issue of constructing appropriately random-Galilean-invariant η^{nl}'s. For practical application of the formulas, however, one must be very careful to use in Eq. (17) the appropriate $\eta^S_{\boldsymbol{k}}$, which is obtained from a formula different from Eq. (16a).

C. Overview of results and comparison to previous work

In general, the mean long-wavelength energy $E_{\boldsymbol{q}}$ evolves according to

$$\partial_t E_{\boldsymbol{q}} = 2(\gamma^{\text{lin}}_{\boldsymbol{q}} + \gamma^{\text{nl}}_{\boldsymbol{q}})E_{\boldsymbol{q}} + \dot{E}^{\text{inc}}_{\boldsymbol{q}}, \tag{21}$$

where $\dot{E}^{\text{inc}}_{\boldsymbol{q}} > 0$. [$\gamma^{\text{nl}}_{\boldsymbol{q}} = -\text{Re}\,\eta^{\text{nl}}_{\boldsymbol{k}}$; cf. Eq. (15).] $\gamma^{\text{nl}}_{\boldsymbol{q}}$ is called the coherent nonlinear response or the nonlinear growth rate; $\dot{E}^{\text{inc}}_{\boldsymbol{q}}$ is called the incoherent response or noise. (For general discussion of coherent and incoherent responses, see Ref. [19].) In this paper we study the contributions to $\gamma^{\text{nl}}_{\boldsymbol{q}}$ and $\dot{E}^{\text{inc}}_{\boldsymbol{q}}$ due to interactions with large $\boldsymbol{k}$'s; we call those contributions $\gamma_{\boldsymbol{q}}$ and $\dot{E}^{\text{noise}}_{\boldsymbol{q}}$. We will show [Eq. (69a) below] that correct to lowest order in ϵ one has, with $\boldsymbol{\nabla} \to i\boldsymbol{q}$,

$$\gamma_{\boldsymbol{q}} = -2\left(\frac{1}{\alpha_{\boldsymbol{q}}+q^2}\right)\sum_{\boldsymbol{k}\ \text{large}} \frac{1}{k^4}(\boldsymbol{k}\cdot\boldsymbol{\nabla}\hat{\Omega}_{\boldsymbol{k};\boldsymbol{q}})^*\,\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}\boldsymbol{\nabla}\hat{\Omega}_{\boldsymbol{k};\boldsymbol{q}}\cdot\frac{\partial Q_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \tag{22a}$$

$$= -2\left(\frac{q^2}{\alpha_{\boldsymbol{q}}+q^2}\right)q^2\sum_{\boldsymbol{k}\ \text{large}}\left(\frac{k_x}{k^4}\right)\hat{\Omega}^2_{\boldsymbol{k}}\,\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}\hat{\boldsymbol{q}}\cdot\frac{\partial Q_{\boldsymbol{k}}}{\partial \boldsymbol{k}}, \tag{22b}$$

where $\theta^r \equiv \text{Re}\,\theta$,

$$Q \doteq \begin{cases} W & (q_\parallel \neq 0) \\ Z & (q_\parallel = 0), \end{cases} \tag{23}$$

and

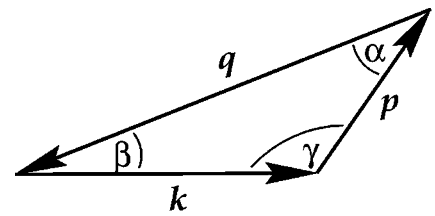

FIG. 1. Geometry for the wave-vector triad such that $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{0}$, with interior angles α, β, and γ. The exterior angles are $\hat{\alpha} \doteq \angle(\boldsymbol{p},\boldsymbol{q}) = \pi - \alpha$, $\hat{\beta} \doteq \angle(\boldsymbol{q},\boldsymbol{k}) = \pi - \beta$, $\hat{\gamma} \doteq \angle(\boldsymbol{k},\boldsymbol{p}) = \pi - \gamma$.

$$\hat{\Omega}_{\boldsymbol{k}} \doteq \left(\frac{k^2}{\alpha_{\boldsymbol{k}}+k^2}\right)k_y = \left(\frac{k^2}{\overline{\overline{k^2}}}\right)k_y, \tag{24a}$$

$$\hat{\Omega}_{\boldsymbol{k};\boldsymbol{q}} \doteq iq\hat{\Omega}_{\boldsymbol{k}}. \tag{24b}$$

In all formulas describing $\gamma_{\boldsymbol{q}}$ and related nonlinear quantities, the x direction is to be interpreted as being parallel to $\boldsymbol{q}$, not to a background profile gradient. Thus $k_x = k\cos\hat{\beta}$ and $k_y = k\sin\hat{\beta}$, where $\hat{\beta}$ is the angle between $\boldsymbol{q}$ and $\boldsymbol{k}$ (see Fig. 1), so $\hat{\Omega}_{\boldsymbol{k}}$ implicitly depends on the direction of $\boldsymbol{q}$. $\hat{\Omega}_{\boldsymbol{k}}$ is an effective $\boldsymbol{E}\times\boldsymbol{B}$ advection frequency based on a unit electric field in the $-\boldsymbol{q}$ direction. Thus, according to Eq. (5), if $\boldsymbol{V}_E \doteq \boldsymbol{E}\times\hat{\boldsymbol{z}}$ were constant and if $\boldsymbol{E} = E\hat{\boldsymbol{q}}$, then the frequency associated with vorticity advection would be $\Omega^{\text{nl}} = -\hat{\boldsymbol{z}}\cdot\boldsymbol{E}\times\boldsymbol{k}\,k^2/(\alpha_{\boldsymbol{k}}+k^2) = -E\hat{\Omega}_{\boldsymbol{k}}$. [Note that the caret has been used in four different ways in the above formulas: as an operator ($\hat{\alpha}$); as a unit vector ($\hat{\boldsymbol{q}}$); as an object from which linear dependence on φ or $\boldsymbol{E}$ has been removed by functional differentiation ($\hat{\Omega}_{\boldsymbol{k}}$); and as an exterior angle ($\hat{\beta}$). Context enables one to distinguish these different usages.] From Eq. (17) and standard reality conditions, one has

$$\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}} = \frac{\text{Re}\,\eta^{\text{nl},S}_{\boldsymbol{q}} + 2\,\text{Re}\,\eta^{\text{nl},S}_{\boldsymbol{k}} - (\gamma^{\text{lin}}_{\boldsymbol{q}} + 2\gamma^{\text{lin}}_{\boldsymbol{k}})}{(\Omega^{\text{lin}}_{\boldsymbol{q}} + \text{Im}\,\eta^{\text{nl},S}_{\boldsymbol{q}})^2 + [\text{Re}\,\eta^{\text{nl},S}_{\boldsymbol{q}} + 2\,\text{Re}\,\eta^{\text{nl},S}_{\boldsymbol{k}} - (\gamma^{\text{lin}}_{\boldsymbol{q}} + 2\gamma^{\text{lin}}_{\boldsymbol{k}})]^2}. \tag{25}$$

(Note that $\gamma^{\text{lin}} = 0$ for pure HM dynamics and that $\Omega^{\text{lin}}_{\boldsymbol{q}} = 0$ for pure zonal flows.) The result obtained in Ref. [2] has the same structural form as Eq. (22b), but with $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$ replaced (in our notation) by

$$\mathcal{R}_{\boldsymbol{k};\boldsymbol{q}} \doteq \frac{2\gamma^{\text{lin}}_{\boldsymbol{k}}}{(\text{Im}\,\eta^{\text{nl}}_{\boldsymbol{q}} - qv^{\text{lin}}_{\text{gr},\boldsymbol{k}})^2 + (2\gamma^{\text{lin}}_{\boldsymbol{k}})^2}, \tag{26}$$

where $v^{\text{lin}}_{\text{gr},\boldsymbol{k}} \doteq \hat{\boldsymbol{x}}\cdot\partial\Omega^{\text{lin}}_{\boldsymbol{k}}/\partial\boldsymbol{k}$ is the linear group velocity. (In Ref. [2], the x direction was in the radial direction opposite to the profile gradient, and $\boldsymbol{q}$ was aligned with that direction because pure zonal flows were considered. That special case unfortunately obscures important differences between linear and nonlinear physics, as we will clarify later.) The several differences between Eqs. (22) and the result of Ref. [2] are seen to be in the form of the effective autocorrelation time between the interacting modes. First, formula (26) contains the wave-packet propagation term $qv^{\text{lin}}_{\text{gr},\boldsymbol{k}}$ in the denominator, whereas the systematically derived result (25) does not. Second, Eq. (26) contains no nonlinear renormalization of the linear growth rate and does not recognize the distinction between η^S and η, whereas our result is fully renormalized and is random Galilean invariant. Third, Eq. (26) is proportional to the linear growth rate $\gamma^{\text{lin}}_{\boldsymbol{k}}$ and changes sign with $\gamma^{\text{lin}}_{\boldsymbol{k}}$, whereas formula (25) is proportional to $\text{Re}\,\eta^S_{\boldsymbol{q}} + 2\,\text{Re}\,\eta^S_{\boldsymbol{k}}$ (the sum of the nonlinear damping rates, *minus* the sum of the linear growth rates) and is intrinsically positive. [Equation (15) guarantees that in steady state $\eta_{\boldsymbol{k}} > 0$, because it is then in balance with the positive-definite covariance $F^{\text{nl}}_{\boldsymbol{k}}$, and a similar result can be obtained for $\eta^S_{\boldsymbol{k}}$.] If one were to merely drop the η's in Eq. (25), one would obtain a form similar to Eq. (26) but with the opposite sign. Of course, formula (26) vanishes for nondissipative HM dynamics, so it is obviously incomplete.

These differences are mostly a consequence of inadequate treatment of renormalization in the derivation of formula (26). (In some subsequent references [31,32], $\gamma_{\boldsymbol{k}}^{\text{lin}}$ was more correctly replaced by a nonlinear frequency spread $\Delta\omega$, for which, however, no formula was given.) The appearance of the linear group velocity of small-scale wave packets in Eq. (26) but not in the systematically derived answer (25) is a more subtle issue that we will discuss in Sec. III C, where we show that $qv_{\text{gr},\boldsymbol{k}}^{\text{lin}}$ is a higher-order effect that must be omitted to lowest order in a systematic ϵ expansion. Indeed, in the course of the article we will show that neither the direct systematic calculations nor our heuristic algorithms explicitly involve any linear properties of the undamped drift wave to lowest order in ϵ.

Upon multiplying Eq. (22a) by $\frac{1}{2}(\alpha_{\boldsymbol{q}}+q^2)\langle|\varphi_{\boldsymbol{q}}|^2\rangle=E_{\boldsymbol{q}}$, one can write it for dimensional purposes as

$$\gamma_{\boldsymbol{q}}E_{\boldsymbol{q}}\sim-\sum_{\boldsymbol{k}\text{ large}}\left\langle\frac{1}{k^2}(|\hat{\gamma}_{\boldsymbol{k};\boldsymbol{q}}^{(1)}|)\left(\theta_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}^{r}|q\hat{\Omega}_{\boldsymbol{k};\boldsymbol{q}}|\hat{\boldsymbol{q}}\cdot\frac{\partial Q_{\boldsymbol{k}}}{\partial\boldsymbol{k}}\right)\right\rangle, \tag{27}$$

where $\hat{\gamma}_{\boldsymbol{k};\boldsymbol{q}}^{(1)}$ is the large-scale Fourier transform ($\boldsymbol{\nabla}\to i\boldsymbol{q}$) of $\hat{\gamma}_{\boldsymbol{k}}^{(1)}\doteq\boldsymbol{k}\cdot\boldsymbol{\nabla}\hat{\Omega}_{\boldsymbol{k};\boldsymbol{q}}/k^2$. The first-order growth rate $\hat{\gamma}_{\boldsymbol{k}}^{(1)}$ describes the rate of evolution of wave number $\boldsymbol{k}$ due to ray propagation in the inhomogeneous $\boldsymbol{q}$ modulation. (For more detailed discussion, see Sec. VI B.) As we will show in Sec. II, the second parenthesized factor in Eq. (27) is the first-order Eulerian enstrophy increment that develops during the effective time $\theta_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}^{r}$ of interaction between modes $\boldsymbol{k}$ and $\boldsymbol{q}$. The product of the two parenthetic factors in Eq. (27) describes how the random first-order distension of the short-wavelength fluctuations leads at second order to a mean energy drain [note the minus sign in Eq. (27)] from the short wavelengths that shows up as long-wavelength growth.

D. Outline

Our goals are several. Given the generalized Hasegawa-Mima model and the Markovian statistical closure formulas described above, we want to systematically obtain (asymptotically to lowest nontrivial order in the scale-separation parameter ϵ) the statistical equations that describe the interactions of disparate scales, we want to interpret the results heuristically, and we want to understand their relationships to prior work. As an introduction, we present in Sec. II a heuristic derivation of $\gamma_{\boldsymbol{q}}$ that emphasizes the role of enstrophy conservation, the significance of $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$, and the intrinsic nonlinearity of the physics. In Sec. III we derive the exact results, including Eqs. (22), as a direct expansion in ϵ of formulas (16) (which contain no assumptions about scale sizes). Although the intermediate algebra is straightforward, it is tedious and without immediate physical interpretation. Therefore, in Sec. IV we present a generalized renormalization procedure that takes explicit account of the presence of disparate scales and provides a systematic apparatus for describing the functional variations of short-wavelength statistics with respect to long-wavelength potentials. As a consistency check, we focus on $\gamma_{\boldsymbol{q}}$ and show that the procedure leads to precisely the same answer as the one obtained in Sec. III. We show in Sec. V that the field-theoretic techniques introduced in Sec. IV can be exploited in an alternative derivation of $\gamma_{\boldsymbol{q}}$ based on a nonlinear statistical energy-balance (Poynting) theorem. We demonstrate that the procedure again leads to the same results as in Secs. III and IV. The general Poynting calculation is not algebraically simpler, but it provides a useful formula that clearly shows how $\gamma_{\boldsymbol{q}}$ arises from second-order variations of the short-wavelength energy. In Sec. VI we show how to use WKB expansion in conjunction with enstrophy conservation to obtain the correct lowest-order answer for $\gamma_{\boldsymbol{q}}$ without the necessity for subsidiary expansion in ϵ. In Secs. VI A and VI B we reexamine the theory of wave kinetic equations, correcting a conceptual mistake in the literature. We describe the physical significance of $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$ in Sec. VI C, then justify the heuristic algorithm of Sec. II in Sec. VI D. In Sec. VI E we show that heuristic Fokker-Planck analysis of the wave-number evolution recovers the dominant wave-number diffusion and drag terms in the spectral evolution equation for the short scales. In Sec. VI F we describe an alternative algorithm based on first-order variation of the enstrophy flux that shows how $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$ is related to an effective nonlinear autocorrelation time. Finally, we conclude with some discussion in Sec. VII. (For further orientation, the reader may find it useful to read the last paragraph of that section, which summarizes the principal results in more detail than we have done so far, before proceeding to the detailed calculations.) The Appendix discusses the relation of our work to a recent alternative calculation of eddy viscosity. A summary of important notation is given in Table I.

II. A HEURISTIC DERIVATION OF THE LONG-WAVELENGTH GROWTH RATE BASED ON SECOND-ORDER ENERGETICS

Before turning to formal calculations, we shall give a heuristic derivation of $\gamma_{\boldsymbol{q}}$ that obtains it from the drain of (second-order) energy from the small scales under the constraint of enstrophy conservation in the face of long-wavelength modulation. For definiteness, we give the details for the pure HM dynamics defined by Eq. (9), but we indicate the ready generalization.

Energy conservation demands that, under the interactions between disparate scales, positive energy increments ΔE in long-wavelength fluctuations (of space scale $\boldsymbol{X}$ and slow time scale T) arise from negative energy increments in the short-scale fluctuations. Thus, with $E^{<}$ ($E^{>}$) denoting the long-(short-)wavelength energy, one has schematically

$$\partial_T\Delta E^{<}=2\gamma^{<}\Delta E^{<}=-\partial_T\Delta E^{>}, \tag{28}$$

where $\gamma^{<}\to\gamma_{\boldsymbol{q}}$ is the nonlinear growth rate. Because the interactions are both nonlinear and random, the variation Δ involves second-order effects. If one notes that the contribution to $E^{<}$ from one Fourier component is $E_{\boldsymbol{q}}=\frac{1}{2}(\alpha_{\boldsymbol{q}}+q^2)\langle|\delta\varphi_{\boldsymbol{q}}|^2\rangle$ and that Eq. (28) contains a factor of 2, a plausible formula (derived in Sec. V) is

TABLE I. Important notation.

Wave numbers	
$\boldsymbol{q}$:	wave vector of long-wavelength fluctuations (e.g., zonal flows). $q=\lvert\boldsymbol{q}_\perp\rvert\leqslant q_{\max}$
$\boldsymbol{k}$:	wave vector of short-wavelength fluctuations (e.g., drift waves). $k=\lvert\boldsymbol{k}_\perp\rvert\geqslant k_{\min}$
k_x, k_y:	$\hat{\boldsymbol{q}}\cdot\boldsymbol{k}$, $\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{q}}\times\boldsymbol{k}$
ϵ:	ordering parameter $q/k\ll 1$
$\overline{k^2}$:	$\alpha_{\boldsymbol k}+k^2$
Potentials	
$\tilde\varphi$:	random potential
P:	mean field $\langle\tilde\varphi\rangle$
$\bar\varphi$:	$k_\parallel=0$ projection of φ
$\check\varphi$:	$k_\parallel\neq 0$ projection of φ
Fluctuation spectra	
$C_{\boldsymbol k}$:	potential spectrum $\langle\lvert\delta\varphi_{\boldsymbol k}\rvert^2\rangle$
$E_{\boldsymbol k}$:	energy spectrum $\sigma^{(E)}_{\boldsymbol k}C_{\boldsymbol k}$ $[\sigma^{(E)}_{\boldsymbol k}\doteq\frac{1}{2}(\alpha_{\boldsymbol k}+k^2)=\frac{1}{2}\overline{k^2}]$
$\mathcal{E}$:	short-wavelength energy $\Sigma_{\boldsymbol k \text{ large}}E_{\boldsymbol k}$
Q:	either W (pure HM) or Z (generalized HM)
$W_{\boldsymbol k}$:	potential enstrophy spectrum $\sigma^{(W)}_{\boldsymbol k}C_{\boldsymbol k}$ $(\sigma^{(W)}_{\boldsymbol k}\doteq k^2\sigma^{(E)}_{\boldsymbol k})$
$Z_{\boldsymbol k}$:	generalized enstrophy spectrum $\sigma^{(Z)}_{\boldsymbol k}C_{\boldsymbol k}$ $(\sigma^{(Z)}_{\boldsymbol k}\doteq\overline{k^2}\sigma^{(E)}_{\boldsymbol k})$
Frequencies	
$\Omega^{\text{lin}}_{\boldsymbol k}$:	HM linear drift-wave frequency $\omega_*/(\alpha_{\boldsymbol k}+k^2)$
$\tilde\Omega_{\boldsymbol k}$:	advection frequency $\hat\Omega_{\boldsymbol k;\boldsymbol q}\tilde\varphi_{\boldsymbol q}$. For pure HM, $\tilde\Omega_{\boldsymbol k}=(k^2/\overline{k^2})\boldsymbol{k}\cdot\boldsymbol{V}_{\boldsymbol q}$; for generalized HM, $\tilde\Omega_{\boldsymbol k}=\boldsymbol{k}\cdot\boldsymbol{V}_{\boldsymbol q}$.
$\hat\Omega_{\boldsymbol k;\boldsymbol q}$:	advection frequency without potential, $iq\hat\Omega_{\boldsymbol k}$
$\hat\Omega_{\boldsymbol k}$:	unit advection frequency, either $(k^2/\overline{k^2})k_y$ or k_y
Growth rates	
$\gamma^{\text{lin}}_{\boldsymbol k}$:	linear growth rate
$\gamma^{(1)}_{\boldsymbol k}$:	first-order wave-number distension rate $\boldsymbol{k}\cdot\boldsymbol{\nabla}\Omega_{\boldsymbol k}/k^2$. $\tilde\gamma^{(1)}_{\boldsymbol k}\doteq\gamma^{(1)}_{\boldsymbol k}[\tilde\Omega_{\boldsymbol k}]$ (brackets denote functional dependence)
$\gamma_{\boldsymbol q}$:	long-wavelength nonlinear growth rate driven by the large $\boldsymbol k$'s

$$\gamma_{\boldsymbol q}=-\left(\frac{1}{\alpha_{\boldsymbol q}+q^2}\right)\left(\frac{\delta^2\partial_T\mathcal{E}}{\delta\tilde\varphi_{\boldsymbol q}\delta\tilde\varphi^*_{\boldsymbol q}}\right)\Bigg|_{\tilde\varphi_{\boldsymbol q}=0}, \tag{29}$$

where $\mathcal{E}\equiv E^>$ is the mean short-wavelength energy $[\mathcal{E}\doteq\frac{1}{2}\Sigma_{\boldsymbol k\text{ large}}(\alpha_{\boldsymbol k}+k^2)\overline{\langle\lvert\delta\varphi_{\boldsymbol k}\rvert^2\rangle}]$ averaged over $\boldsymbol X$ (denoted by the overbar) and the tilde indicates the random nature of $\tilde\varphi_{\boldsymbol q}$. (In the formal work of subsequent sections, we will differentiate instead with respect to the mean field $P_{\boldsymbol q}\doteq\langle\varphi_{\boldsymbol q}\rangle$.) Now Smolyakov and Diamond [15] have shown that for pure HM dynamics the enstrophy density $W_{\boldsymbol k}\doteq k^2E_{\boldsymbol k}$ is conserved under a long-wavelength modulation. Upon concentrating only on nonlinear contributions to $W_{\boldsymbol k}$, one has the wave kinetic equation (WKE)

$$\partial_T W_{\boldsymbol k}(\boldsymbol X,T)+\tilde{\boldsymbol V}_{\text{gr},\boldsymbol k}\cdot\boldsymbol\nabla W_{\boldsymbol k}-\boldsymbol\nabla\tilde\Omega_{\boldsymbol k}\cdot\partial_{\boldsymbol k}W_{\boldsymbol k}=0, \tag{30}$$

where $\boldsymbol\nabla\equiv\partial/\partial\boldsymbol X$, $\partial_{\boldsymbol k}\equiv\partial/\partial\boldsymbol k$, and

$$\tilde\Omega_{\boldsymbol k}(\boldsymbol X,T)\doteq\left(\frac{k^2}{\alpha_{\boldsymbol k}+k^2}\right)\boldsymbol k\cdot\hat{\boldsymbol z}\times\boldsymbol\nabla\tilde{\bar\varphi} \tag{31}$$

is the random advection frequency associated with the long-wavelength potential; $\tilde{\boldsymbol V}_{\text{gr},\boldsymbol k}\doteq\partial\tilde\Omega_{\boldsymbol k}/\partial\boldsymbol k$ is the associated group velocity (*not* the linear group velocity). For considering the total short-wavelength enstrophy or energy, the equivalent equation

$$\partial_T W_{\boldsymbol k}(\boldsymbol X,T)+\boldsymbol\nabla\cdot(\tilde{\boldsymbol V}_{\text{gr},\boldsymbol k}W_{\boldsymbol k})-\partial_{\boldsymbol k}\cdot(\boldsymbol\nabla\tilde\Omega_{\boldsymbol k}W_{\boldsymbol k})=0 \tag{32}$$

is more useful, as it is in conservation form. Upon averaging Eq. (32) over space, one finds that the mean short-scale enstrophy evolves according to

$$\partial_T\overline{W}_{\boldsymbol k}(T)=\partial_{\boldsymbol k}\cdot\overline{(\boldsymbol\nabla\tilde\Omega_{\boldsymbol k}W_{\boldsymbol k})}. \tag{33}$$

Parseval's theorem can be used to replace the barring operation by a sum over all $\boldsymbol q$'s: for arbitrary functions $a(\boldsymbol X)$ and $b(\boldsymbol X)$, one has $\overline{a(\boldsymbol X)b(\boldsymbol X)}=\Sigma_{\boldsymbol q}a^*_{\boldsymbol q}b_{\boldsymbol q}$. This might appear to involve large $\boldsymbol q$'s as well as small ones, but since we have restricted the wave kinetic dynamics to first-order gradients, q is effectively small. Thus, in subsequent formulas we shall write $\Sigma_{\boldsymbol q\text{ small}}\to\Sigma_{\boldsymbol q}$ and similarly $\Sigma_{\boldsymbol k\text{ large}}\to\Sigma_{\boldsymbol k}$. Then enstrophy variations due to long-wavelength modulations at wave vectors $\boldsymbol q$ obey

$$\partial_T\overline{W}_{\boldsymbol k}=\frac{\partial}{\partial\boldsymbol k}\cdot\sum_{\boldsymbol q}(i\boldsymbol q\tilde\Omega_{\boldsymbol k;\boldsymbol q})^*W_{\boldsymbol k;\boldsymbol q}, \tag{34}$$

where

$$\tilde\Omega_{\boldsymbol k;\boldsymbol q}\doteq iq\hat\Omega_{\boldsymbol k}\tilde\varphi_{\boldsymbol q}. \tag{35}$$

(Here and subsequently we write $\tilde\varphi_{\boldsymbol q}$ instead of the redundant $\tilde{\bar\varphi}_{\boldsymbol q}$.) An equation for the short-wavelength energy $\mathcal E$ follows by dividing Eq. (34) by k^2 and summing the result over the large $\boldsymbol k$'s:

$$\partial_T\mathcal E=\sum_{\boldsymbol k}\sum_{\boldsymbol q}\left(\frac{1}{k^2}\right)\left(-i\boldsymbol q\cdot\frac{\partial}{\partial\boldsymbol k}(\tilde\Omega^*_{\boldsymbol k;\boldsymbol q}W_{\boldsymbol k;\boldsymbol q})\right) \tag{36a}$$

$$=i\sum_{\boldsymbol k}\sum_{\boldsymbol q}\left[\boldsymbol q\cdot\frac{\partial}{\partial\boldsymbol k}\left(\frac{1}{k^2}\right)\right]\tilde\Omega^*_{\boldsymbol k;\boldsymbol q}W_{\boldsymbol k;\boldsymbol q} \tag{36b}$$

$$=-2q^2\sum_{\boldsymbol k}\sum_{\boldsymbol q}\left(\frac{k_x}{k^4}\right)\hat\Omega_{\boldsymbol k}\tilde\varphi^*_{\boldsymbol q}W_{\boldsymbol k;\boldsymbol q}. \tag{36c}$$

We emphasize again that all Cartesian directions in these formulas are relative to $\boldsymbol q$. To arrive at Eq. (36b), we integrated the last term of Eq. (36a) by parts and ignored the associated surface term (which describes interactions with moderately sized $\boldsymbol k$'s). Upon performing the two variations required by formula (29), one obtains

$$\gamma_{\boldsymbol q}=2\left(\frac{q^2}{\alpha_{\boldsymbol q}+q^2}\right)\sum_{\boldsymbol k}\left(\frac{k_x}{k^4}\right)\hat\Omega_{\boldsymbol k}\hat W_{\boldsymbol k;\boldsymbol q}. \tag{37}$$

The first-order enstrophy variation $\hat W_{\boldsymbol k;\boldsymbol q}\doteq\delta W_{\boldsymbol k}/\delta\tilde\varphi_{\boldsymbol q}$ follows from the Fourier transform of Eq. (30); at $\tilde\varphi_{\boldsymbol q}=0$,

$$\partial_T \hat{W}_{\boldsymbol{k};\boldsymbol{q}} = -(\hat{\boldsymbol{V}}_{\mathrm{gr},\boldsymbol{k};\boldsymbol{q}} \cdot \underline{\boldsymbol{\nabla} W_{\boldsymbol{k}}} - i\hat{\Omega}_{\boldsymbol{k};\boldsymbol{q}} \cdot \partial_{\boldsymbol{k}} W_{\boldsymbol{k}}). \tag{38}$$

For spatially homogeneous statistics, unperturbed fluctuation spectra are independent of $\boldsymbol{X}$, so the underlined term on the right-hand side of Eq. (38) vanishes. Under the modulation, the final enstrophy variation builds up over the mode-mode interaction time $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$, so integrating Eq. (38) over that time yields

$$\hat{W}_{\boldsymbol{k};\boldsymbol{q}} = -q^2 \hat{\Omega}_{\boldsymbol{k}} \theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}} \hat{\boldsymbol{q}} \cdot \partial_{\boldsymbol{k}} W_{\boldsymbol{k}}. \tag{39}$$

Upon inserting this result into Eq. (37), one finds Eq. (22b). One of the significant results of the formal calculations in Secs. III–V is the justification of $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$ as the relevant interaction time.

This calculation can be repeated for the generalized dynamics described by Eq. (7b). The minor changes are to write Eq. (30) for Z instead of W, use $\tilde{\Omega}_{\boldsymbol{k}} = \boldsymbol{k} \cdot \tilde{\boldsymbol{V}}_E$ instead of Eq. (31), and divide the Z analog of Eq. (34) by $\overline{k^2} = Z_{\boldsymbol{k}}/E_{\boldsymbol{k}}$ instead of k^2. One is led again to Eq. (22b). [Note that $k_y/\overline{k^2} = \hat{\Omega}_{\boldsymbol{k}}/k^2$ with $\hat{\Omega}_{\boldsymbol{k}}$ defined by Eq. (24a).]

Formula (29) can be criticized on the grounds that it purports to calculate the statistical property $\gamma_{\boldsymbol{q}}$ by examining variations of the averaged quantity $\mathcal{E}$ with respect to *random* potentials. In fact, the proper formula that we will derive later [Eq. (137)] involves variations with respect to the *mean* potential $P_{\boldsymbol{q}}$. Now the statistical problem defined by Eq. (6) is assumed to be spatially homogeneous, in which case $P_{\boldsymbol{q}} = 0$. However, we will see that the correct generalization of formula (29) requires the second-order response to the presence of a *nonzero* mean potential $P_{\boldsymbol{q}}$, perhaps introduced by an external symmetry-breaking perturbation to Eq. (6) or by averaging in an inhomogeneous statistical subensemble. We will make these notions mathematically precise in Sec. IV. However, the distinction between variations with respect to $\tilde{\varphi}_{\boldsymbol{q}}$ and $P_{\boldsymbol{q}}$ is moot at lowest order in ϵ. Further discussion about the subtly different roles of $\tilde{\varphi}_{\boldsymbol{q}}$ and $P_{\boldsymbol{q}}$ is given in Sec. VI.

This algorithm highlights the importance of wave-number evolution due to the slightly inhomogeneous advection frequency. Note that the physics content of the last term of Eq. (38), which figured prominently in the derivation, is the ray equation $d\boldsymbol{k}/dT = -\boldsymbol{\nabla}\tilde{\Omega}_{\boldsymbol{k}}$. We will discuss the significance of this equation more systematically in Sec. VI, where we use WKB and Fokker-Planck techniques to derive various WKE's for energy and enstrophy evolution. The present algorithm is justified in Sec. VI D, and an alternative algorithm based on first-order flux variation is presented in Sec. VI F. All of the heuristic procedures as well as the formal derivations show that the relevant physics has little to do with linear dispersion relations or normal modes; the results depend only on the properties of the nonlinear advection except for trivial linear dissipative effects on the value of the triad interaction time. We will return to this important point at various places throughout the subsequent discussion.

III. FORMAL STATISTICAL CALCULATIONS OF THE INTERACTIONS BETWEEN DISPARATE SCALES

In this and the next section we show how to obtain Eqs. (22) and related formulas by using well-known results and techniques of renormalized statistical dynamics [19]. The general goal is to obtain the contributions to Eq. (15) for $\partial_t C_{\boldsymbol{k}}$ from fluctuations with $q \ll k$, and the corresponding (energy-conserving) terms in the equation for $\partial_t C_{\boldsymbol{q}}$. We will obtain those contributions by successively employing several different approaches. In the first one, which we call the direct method and describe in the present section, we simply expand formulas (16) in the limit $\epsilon \ll 1$. Given the standard assumptions built into second-order statistical closure, the direct method is exact to the order retained; it invokes no subsidiary physical or mathematical assumptions. In subsequent sections we consider alternative approaches. We verify that all procedures lead to consistent results.

Note that Eq. (15) is written for $C_{\boldsymbol{k}} \doteq \langle |\delta\varphi_{\boldsymbol{k}}|^2 \rangle$, whose sum over $\boldsymbol{k}$ is not conserved by the nonlinear interactions. Although the content of the final results cannot depend on the choice of dependent variable, a particular choice may simplify intermediate algebra. For pure HM dynamics, Kraichnan's results as well as the work of Ref. [15] motivate consideration of the equation for $\partial_t W_{\boldsymbol{k}}$. For generalized HM, the work of Ref. [15] suggests that consideration of $\partial_t Z_{\boldsymbol{k}}$ is useful. Upon eliminating the factor of $M^*_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$ in Eq. (16b) (our mode-coupling coefficients are real, so we shall subsequently drop the complex conjugate on the M's) by using one of the detailed conservation laws (11), one is led, for any nonlinear invariant Q, to

$$\partial_t Q_{\boldsymbol{k}} = 2\sum_{\Delta} M^{(Q)}_{\boldsymbol{k}} M^{(Q)}_{\boldsymbol{p}} \theta^{(Q)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} Q_{\boldsymbol{q}} (Q_{\boldsymbol{k}} - Q_{\boldsymbol{p}}), \tag{40}$$

where

$$M^{(Q)}_{\boldsymbol{k}} \doteq \sigma^{(Q)}_{\boldsymbol{k}} M_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}, \tag{41a}$$

$$\theta^{(Q)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \doteq \mathrm{Re}\, \theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} / \sigma^{(Q)}_{\boldsymbol{k}} \sigma^{(Q)}_{\boldsymbol{p}} \sigma^{(Q)}_{\boldsymbol{q}}. \tag{41b}$$

The conservation of Q is immediate, given the antisymmetry of the summand of Eq. (40) in $\boldsymbol{k}$ and $\boldsymbol{p}$.

For future use, we record that Eq. (40) is explicitly

$$\partial_t Z_{\boldsymbol{k}} = \frac{1}{2} \sum_{\Delta} (\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q})^2 [\overline{k^2}(\overline{q^2} - \overline{p^2})][\overline{p^2}(\overline{k^2} - \overline{q^2})] \times \theta^{(Z)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} Z_{\boldsymbol{q}} (Z_{\boldsymbol{k}} - Z_{\boldsymbol{p}}) \tag{42a}$$

$$= \frac{1}{2} \sum_{\Delta} p^2 \sin^2\alpha \left(\frac{q^2}{\overline{q^2}} \right) (\overline{q^2} - \overline{p^2})(\overline{k^2} - \overline{q^2}) \times \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} Z_{\boldsymbol{q}} (Z_{\boldsymbol{k}} - Z_{\boldsymbol{p}}), \tag{42b}$$

where

$$\theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} = \frac{8\,\theta^r_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}}{(\alpha_{\boldsymbol{k}} + k^2)(\alpha_{\boldsymbol{p}} + p^2)(\alpha_{\boldsymbol{q}} + q^2)} \tag{43}$$

and the triad geometry is shown in Fig. 1. The role of the factor $(q^2/\overline{q^2})$ in Eq. (42b) is to change $Z_{\boldsymbol{q}}$ to $W_{\boldsymbol{q}}$. Note that the law of sines enables one to employ alternative angles in Eq. (42b) if that is convenient; e.g., $p\sin\alpha = k\sin\beta$.

Equation (42b) has the form

$$\partial_t Z_{\boldsymbol{k}} = \sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},0} K(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) \tag{44a}$$

$$= \sum_{\boldsymbol{q}} K(\boldsymbol{k},-(\boldsymbol{k}+\boldsymbol{q}),\boldsymbol{q}), \tag{44b}$$

where $\delta_{\boldsymbol{k},\boldsymbol{k}'}$ is the Kronecker delta function and K is a known function. The form (44b) can be expanded directly in ϵ, as we will do in Sec. III B, leading to results valid for arbitrary anisotropy. However, it is instructive to first assume isotropy in $\boldsymbol{k}_\perp$ (so that $Z_{\boldsymbol{k}}$ and $\theta^{(Z)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$ depend only on wave-number magnitude). The algebra of the isotropic and anisotropic calculations is sufficiently different that reconciling the results provides a nontrivial check on the calculations, and the isotropic calculations illustrate certain issues about wave-number integration domains with a minimum of complication.

A. Isotropic calculations

To avoid some clutter in the notation, we shall present the isotropic calculations for pure HM dynamics, so we write W rather than Z in this section. We also follow the original HM assumption and assume that the dynamics are local in $\boldsymbol{x}_\perp$, so we do not sum over $k_\parallel$. The generalization to 3D fluctuations is obvious, and the fully anisotropic algebra is sketched in Sec. III B. For isotropic statistics, angle dependence arises only from the δ function in Eq. (44a) because the mode-coupling coefficients can be written entirely in terms of wave-number magnitudes with the aid of the law of cosines. To integrate over angle, we shall pass to the continuum limit. A consistent set of Fourier transform conventions in a box of length L in spatial dimensionality d is, for the discrete transform,

$$A_{\boldsymbol{k}} = \frac{1}{L^d}\int_{-L/2}^{L/2} d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} A(\boldsymbol{x}), \quad A(\boldsymbol{x}) = \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} A_{\boldsymbol{k}}, \tag{45}$$

and, for the continuous transform,

$$A(\boldsymbol{k}) = \int_{-\infty}^{\infty} d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} A(\boldsymbol{x}), \quad A(\boldsymbol{x}) = \int_{-\infty}^{\infty} \frac{d\boldsymbol{k}}{(2\pi)^d} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} A(\boldsymbol{k}). \tag{46}$$

One passes between the discrete and continuous representations with the replacements

$$\sum_{\boldsymbol{k}} \to \int_{-\infty}^{\infty} \frac{d\boldsymbol{k}}{\delta k^d}, \quad A_{\boldsymbol{k}} \to L^{-d} A(\boldsymbol{k}), \quad \delta_{\boldsymbol{k},0} \to \delta k^d\, \delta(\boldsymbol{k}), \tag{47}$$

where $\delta k \doteq 2\pi/L$. A consequence is that $C_{\boldsymbol{k}} \doteq \langle |A_{\boldsymbol{k}}|^2\rangle \to L^{-d} C(\boldsymbol{k})$, where $C(\boldsymbol{k}) \doteq \int d\boldsymbol{\rho}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}} \langle A(\boldsymbol{x}+\boldsymbol{\rho}) A(\boldsymbol{x})\rangle$. It is then a standard result that when the angular integrations are performed in 2D one finds

$$\sum_{\Delta} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}} \to \frac{1}{N}\int_{\Delta} dp\, dq\, \frac{2}{|\sin\alpha|}, \tag{48}$$

where $N \doteq \delta k^d$ and the integration domain Δ is shown in Fig. 2. Upon inserting this result into Eq. (42b), one obtains

$$\partial_t W_k = \frac{1}{N}\int_{\Delta} dp\, dq |\sin\alpha| (q^2-p^2) p^2 (k^2-q^2) \times \theta^{(E)}_{k,p,q} W_q (W_k - W_p), \tag{49}$$

with the law of cosines giving

$$|\sin\alpha| = (2pq)^{-1}[-(p^2-q^2)^2 + 2(p^2+q^2)k^2 - k^4]^{1/2}. \tag{50}$$

1. Evolution of large k's (isotropic spectrum)

Consider the contributions to formula (49) from fluctuations with $q \ll k$ (region A) or $p \ll k$ (region B); see Fig. 3. For region A, write $p = k + \lambda q$, where $-1 \leqslant \lambda \leqslant 1$. Then $|\sin\alpha| = (1-\lambda^2)^{1/2}(1-\frac{1}{2}\lambda q/k) + O(\epsilon^2)$. Also define $F_{kpq} \doteq \theta^{(E)}_{k,p,q}(W_p - W_k)$ and expand in small q to find

$$F_{kpq} = \lambda q \left(\frac{\partial F}{\partial p}\right)\bigg|_k + \frac{1}{2}(\lambda q)^2 \left(\frac{\partial^2 F}{\partial p^2}\right)\bigg|_k + O(\epsilon^3). \tag{51}$$

Then to dominant order one finds

$$\partial_t W_k^{(\mathrm{A})} = \frac{1}{2} k^4 \left(\int_{-1}^{1} d\lambda \sqrt{1-\lambda^2}\,\lambda^2\right) \times \frac{1}{N}\int_0^{q_{\max}} q\, dq\, q^2 W_q \theta^{(E)}_{k,k,q} \left(7k \frac{\partial F}{\partial p}\bigg|_k + k^2 \frac{\partial^2 F}{\partial p^2}\bigg|_k\right). \tag{52}$$

Further algebra using the definition of F and the result $(\partial\theta^{(E)}_{k,p,q}/\partial p)|_k = \frac{1}{2}\partial\theta^{(E)}_{k,k,q}/\partial k$ leads one to

$$\partial_t W_k^{(\mathrm{A})} = \frac{1}{k}\frac{\partial}{\partial k}\left(k D_k \frac{\partial W_k}{\partial k}\right), \tag{53}$$

where

$$D_k \doteq \frac{1}{4} I k^6 \frac{1}{N}\int_0^{q_{\max}} 2\pi q\, dq\, q^2 W_q \theta^{(E)}_{k,k,q} \tag{54}$$

with $I \equiv I_{2,2} = \frac{1}{8}$ and $I_{m,n} \doteq (2\pi)^{-1}\int_0^{2\pi} d\theta \sin^m\theta \cos^n\theta$. We have written the diffusion operator in the natural form of a divergence in (an isotropic) k space. It conserves W except for a boundary term (which describes the flow of enstrophy to the intermediate scales).

Region B can be treated symmetrically by writing $q = k + \lambda p$, with $|\sin\alpha| \approx (1-\lambda^2)^{1/2}(1-\frac{1}{2}\lambda p/k)$. It is not hard to verify that region B contributes at one higher order in $(q_{\max}/k)^2$. The $W_q W_k$ term is related to incoherent noise

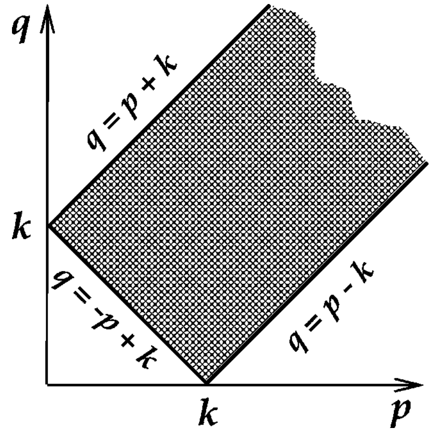

FIG. 2. Integration domain Δ for all wave-number magnitudes p and q such that $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}$ for fixed $\boldsymbol{k}$.

(note that q is large in region B); the remaining terms, together with higher-order contributions from region A, merely provide corrections to the wave-number diffusion effect. After some algebra, one finds

$$\partial_t W_k^{(\mathrm{B})} \approx -\frac{1}{k}\frac{\partial}{\partial k}(kV_k W_k), \tag{55}$$

where

$$V_k \doteq \frac{1}{2} I k^3 \left(\frac{1}{N} \int_0^{q_{\max}} 2\pi p\, dp\, p^4 \theta^{(E)}_{k,p,k} \right) W_k . \tag{56}$$

The drag operator in Eq. (55) also conserves W except for a boundary term.

2. Evolution of small q's (isotropic spectrum)

Now consider the evolution of the small q's with the isotropic assumption. One must integrate over regions C and D in Fig. 4. For region C, write $p=k+\lambda q$; then $|\sin\gamma| =(q/k)(1-\lambda^2)^{1/2}(1-\frac{1}{2}\lambda q/k)+O(\epsilon^2)$. After tedious but straightforward algebra, one finds that to lowest order

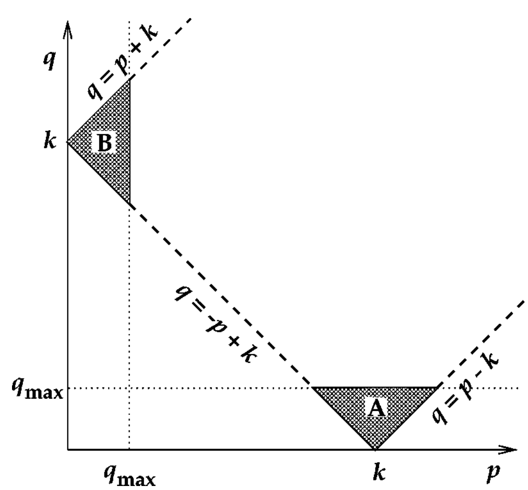

FIG. 3. Integration domain for the effect of small wave vectors $[q \ll k$ (region A) or $p \ll k$ (region B)$]$ on a large wave vector $\boldsymbol{k}$.

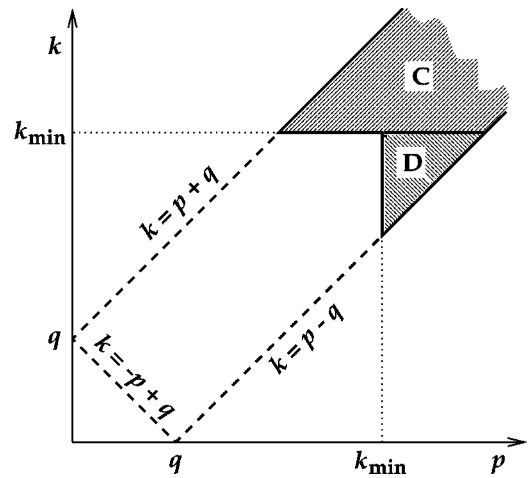

FIG. 4. Integration domain for the interaction of large wave vectors $\boldsymbol{k}$ and $\boldsymbol{p}$ for fixed, small $\boldsymbol{q}$. Region C: $k \geqslant k_{\min}$. Region D: $k<k_{\min}$, $p \geqslant k_{\min}$.

$$\partial_t W_q^{(\mathrm{C})} = -\frac{1}{2} I q^4 \frac{1}{N}\Bigg(\int_{k_{\min}}^{\infty} 2\pi k\, dk\, k^3 \theta^{(E)}_{q,k,k} W'_k W_q + 2\pi[k^4 \theta^{(E)}_{q,k,k} W_k (W_q - W_k)]\big|_{k_{\min}} \Bigg), \tag{57}$$

where $W'_k \doteq \partial W/\partial k$ and the last term is a surface contribution arising after an integration by parts. (For a more explicit calculation, see Sec. III B 2.) To treat region D, define $\tau \doteq p-k$, $T \doteq \frac{1}{2}(p+k)$, then write $\tau=\lambda k$ $(0 \leqslant \lambda \leqslant 1)$ and $T = k_{\min}+\lambda\rho q$ $(-\frac{1}{2} \leqslant \rho \leqslant \frac{1}{2})$. One has $|\sin\gamma| \approx (q/k_{\min})(1-\lambda^2)^{1/2}$. In evaluating formula (49), one may replace p and k by $k_{\min}$ everywhere except in $\sin\gamma$ and in $p^2-k^2 \approx 2\lambda q k_{\min}$. One then finds that the contribution from region D precisely cancels the surface term in Eq. (57), giving rise to the growth rate γ_q given by Eq. (62) below.

Finally, to obtain the contribution of incoherent noise to the small k's, symmetrize the $W_p W_q$ term in Eq. (49) to find, after interchanging k and q,

$$\dot{W}_q^{\mathrm{noise}} = \frac{1}{2} q^2 \frac{1}{N} \int_{\Delta} dp\, dk |\sin\gamma| (p^2-k^2)^2 \theta^{(E)}_{q,k,k} W_p W_k \tag{58a}$$

$$\approx I q^6 \frac{1}{N} \int_{k_{\min}}^{\infty} 2\pi k\, dk\, \theta^{(E)}_{q,k,k} W_k^2 . \tag{58b}$$

In summary of the isotropic calculations, we have found that for pure HM dynamics the small scales evolve under their interaction with the large ones according to

$$\partial_t W_k = \frac{1}{k}\frac{\partial}{\partial k}\left(k D_k \frac{\partial W_k}{\partial k} \right) - \frac{1}{k}\frac{\partial}{\partial k}(kV_k W_k), \tag{59}$$

where

$$D_k \doteq \frac{1}{4} I k^6 \frac{1}{N} \int_0^{q_{\max}} 2\pi q\, dq\, q^2 \theta^{(E)}_{k,k,q} W_q , \tag{60a}$$

$$V_k \doteq \frac{1}{2} I k^3 \left(\frac{1}{N} \int_0^{q_{\max}} 2\pi p\, dp\, p^4 \theta^{(E)}_{k,p,k} \right) W_k , \qquad (60b)$$

and that the corresponding long-wavelength evolution is

$$\partial_t W_q = 2\gamma_q W_q + \dot{W}_q^{\text{noise}}, \qquad (61)$$

where

$$\gamma_q \doteq -\frac{1}{4} q^4 I \frac{1}{N} \int_{k_{\min}}^{\infty} 2\pi k\, dk\, k^3 \theta^{(E)}_{q,k,k} W'_k \qquad (62)$$

and $\dot{W}_q^{\text{noise}}$ is defined by Eq. (58b). To verify that these equations appropriately conserve energy, divide Eq. (59) by k^2, then integrate $\int_{k_{\min}}^{\infty} k\, dk$. Also, divide Eq. (61) by q^2, then integrate $\int_0^{q_{\max}} q\, dq$. It is straightforward to verify that the energy drain from the wave-number diffusion is accounted for by the γ_q term, and that the energy drain from the V_k drag shows up in $\dot{W}_q^{\text{noise}}$.

As a check, one can compare Eq. (62) with Kraichnan's eddy viscosity for 2D isotropic Navier-Stokes turbulence. With $\nu(q|k_{\min}) \doteq -\gamma_q/q^2$, and upon recalling the definition (43) of $\theta^{(E)}$, one can write Eq. (62) more explicitly as

$$\nu_{\text{HM}}(q|k_{\min}) = \frac{\pi}{2} \left(\frac{q^2}{\alpha_q + q^2} \right) \times \frac{1}{N} \int_{k_{\min}}^{\infty} dk \left(\frac{k^2}{\alpha_k + k^2} \right)^2 \theta^r_{q,k,k} \frac{dW_k}{dk}. \qquad (63)$$

One passes from HM to NS dynamics by setting $\hat{\alpha} = 0$. Kraichnan writes $E_k = \frac{1}{2} \delta k^d U(k)$ [see his Eq. (2.8)]. Then

$$\nu_{\text{NS}}(q|k_{\min}) = \frac{\pi}{4} \int_{k_{\min}}^{\infty} dk\, \theta^r_{q,k,k} \frac{d(k^2 U)}{dk}. \qquad (64)$$

This agrees exactly with Kraichnan's Eq. (4.6).

Kraichnan gave a thorough discussion of the physical mechanism (involving W conservation) underlying this result, and his insights carry over to HM dynamics without essential change. A key observation is that, to the extent that $\theta^r_{q,k,k}$ is a function θ^r_q independent of k, as it will be when the interactions are dominated by large-scale random straining, formula (64) involves a perfect derivative and is negative. In this same approximation, formula (63) is negative as well, as follows from integration by parts:

$$\nu_{\text{HM}} = -\frac{\pi}{2} \left(\frac{q^2}{\alpha_q + q^2} \right) \theta^r_q \frac{1}{N} \left[\left(\frac{k^4 W_k}{(\alpha_k + k^2)^2} \right) \Bigg|_{k_{\min}} + 4\alpha_k \int_{k_{\min}}^{\infty} dk \frac{k^2 W_k}{(\alpha_k + k^2)^3} \right]. \qquad (65)$$

These results do not require that dW_k/dk be uniformly negative; it is merely necessary that W_k vanish at ∞. Further discussion of Eq. (65) and its relation to the results of Ref. [7] is given in the Appendix.

B. Anisotropic calculations

We now turn to the general anisotropic case, which we present for generalized HM dynamics. For short-scale evolution, one finds the Fokker-Planck equation

$$\partial_t Z_{\boldsymbol{k}} = \frac{\partial}{\partial \boldsymbol{k}} \cdot \left(\mathsf{D}_{\boldsymbol{k}} \cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right) - \frac{\partial}{\partial \boldsymbol{k}} \cdot (V_{\boldsymbol{k}} Z_{\boldsymbol{k}}), \qquad (66)$$

where

$$\mathsf{D}_{\boldsymbol{k}} \doteq \frac{1}{4} \overline{k^4} \sum_{\boldsymbol{q}} k_y^2 q^2 W_{\boldsymbol{q}} \theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}} (\hat{\boldsymbol{q}} \hat{\boldsymbol{q}}), \qquad (67a)$$

$$V_{\boldsymbol{k}} \doteq \frac{1}{2} \left(\sum_{\boldsymbol{q}} \hat{\boldsymbol{q}} k_y^2 k_x q^4 \theta^{(E)}_{\boldsymbol{k},\boldsymbol{q},-\boldsymbol{k}} \right) Z_{\boldsymbol{k}}. \qquad (67b)$$

For the long-wavelength evolution, one finds

$$\partial_t Z_{\boldsymbol{q}} = 2\gamma_{\boldsymbol{q}} Z_{\boldsymbol{q}} + \dot{Z}_{\boldsymbol{q}}^{\text{noise}}, \qquad (68)$$

where

$$\gamma_{\boldsymbol{q}} \doteq -\frac{1}{4} q^4 \sum_{\boldsymbol{k}} k_y^2 k_x \theta^{(E)}_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}} \hat{\boldsymbol{q}} \cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}}, \qquad (69a)$$

$$\dot{Z}_{\boldsymbol{q}}^{\text{noise}} \doteq \overline{q^2} q^4 \sum_{\boldsymbol{k}} \left(\frac{k_x^2 k_y^2}{\overline{k^4}} \right) \theta^{(E)}_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}} Z_{\boldsymbol{k}}^2. \qquad (69b)$$

It is easily verified that these forms reduce properly to the isotropic HM results summarized in Eqs. (59)–(62) by dropping the overbars on k and q [denominators of $(1+k^2)$ are still retained in the $\theta^{(E)}$'s], replacing Z by W, and noting that for isotropic statistics $\hat{\boldsymbol{q}} \cdot \partial W_{\boldsymbol{k}} / \partial \boldsymbol{k} = (k_x/k) W'_k$ and the $\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}$ component of D is proportional to $k_x^2 k_y^2$. Note that Eq. (69a) can be written in the forms (22) by recalling the definitions (43) of $\theta^{(E)}$ and (24a) of $\hat{\Omega}_{\boldsymbol{k}}$. It is also straightforward to verify the energy-conservation theorems

$$-\sum_{\boldsymbol{k}} \frac{1}{\overline{k^2}} \frac{\partial}{\partial \boldsymbol{k}} \cdot \left(\mathsf{D}_{\boldsymbol{k}} \cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right) = \sum_{\boldsymbol{q}} \frac{1}{\overline{q^2}} (2\gamma_{\boldsymbol{q}} Z_{\boldsymbol{q}}), \qquad (70a)$$

$$\sum_{\boldsymbol{k}} \frac{1}{\overline{k^2}} \frac{\partial}{\partial \boldsymbol{k}} \cdot (V_{\boldsymbol{k}} Z_{\boldsymbol{k}}) = \sum_{\boldsymbol{q}} \frac{1}{\overline{q^2}} \dot{Z}_{\boldsymbol{q}}^{\text{noise}}. \qquad (70b)$$

We emphasize that Eqs. (66) and (68) are not the complete spectral balance equations; they describe only the interactions due to disparate scales.

In the following subsections, we describe the derivations of these formulas. We shall present the algebra for the $\mathsf{D}_{\boldsymbol{k}}$ term in some detail, since it illustrates some tricky points. We merely sketch the remaining calculations.

1. Evolution of large k's (anisotropic spectrum)

To derive formula (66), one must consider interactions of triads with shapes shown in Fig. 5. For region A (q small), Eq. (42b) is conveniently written with the aid of the law of sines as

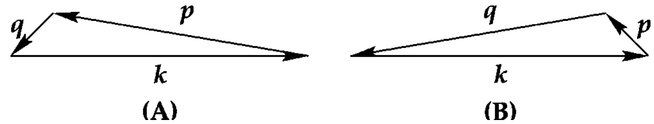

FIG. 5. Possible triads with one small leg that contribute to the evolution of disparate scales. Region A: q small; region B: p small.

$$\partial_t Z_{\boldsymbol{k}}^{(\mathrm{A})} = \frac{1}{2} k^2 \sum_{\boldsymbol{q}} s^2 Z_{\boldsymbol{q}} \left(\frac{q^2}{\overline{q^2}} \right) (\overline{p^2} - \overline{q^2})(\overline{k^2} - \overline{q^2}) F_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}, \tag{71}$$

where $s \doteq \sin\beta$ and $F_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \doteq \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}(Z_{\boldsymbol{p}} - Z_{\boldsymbol{k}})$. Since $\boldsymbol{p} = -\boldsymbol{k} - \boldsymbol{q}$, one has $p^2 = k^2 + 2\boldsymbol{k}\cdot\boldsymbol{q} + q^2$. Also, one may expand $F_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$ around $\boldsymbol{p} = -\boldsymbol{k}$:

$$F_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} = -\boldsymbol{q}\cdot \left(\frac{\partial F}{\partial \boldsymbol{p}} \right)\bigg|_{-\boldsymbol{k}} + \frac{1}{2}(\boldsymbol{q}\boldsymbol{q}) : \left(\frac{\partial^2 F}{\partial \boldsymbol{p} \partial \boldsymbol{p}} \right)\bigg|_{-\boldsymbol{k}} + O(\epsilon^3), \tag{72}$$

where the lowest-order term $F_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}}$ vanished by definition of F. (Note that $Z_{-\boldsymbol{k}} = Z_{\boldsymbol{k}}$.) Upon retaining terms through $O(q^2)$ and recalling that $(q^2/\overline{q^2})Z_{\boldsymbol{q}} = W_{\boldsymbol{q}}$, one finds to dominant order

$$\partial_t Z_{\boldsymbol{k}}^{(\mathrm{A})} = \frac{1}{2} k^2 \sum_{\boldsymbol{q}} s^2 W_{\boldsymbol{q}} q^2 \left(-2\overline{k^2}\boldsymbol{k}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\cdot \frac{\partial F}{\partial \boldsymbol{p}} + \frac{1}{2}\overline{k^4}(\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) : \frac{\partial^2 F}{\partial \boldsymbol{p}\, \partial \boldsymbol{p}} \right), \tag{73}$$

where to avoid clutter we no longer explicitly indicate that $\boldsymbol{p}$ is to be replaced by $-\boldsymbol{k}$. Now

$$\frac{\partial F}{\partial \boldsymbol{p}} = \frac{\partial \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}}{\partial \boldsymbol{p}} (Z_{\boldsymbol{p}} - Z_{\boldsymbol{k}}) + \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \frac{\partial Z_{\boldsymbol{p}}}{\partial \boldsymbol{p}}, \tag{74a}$$

$$\frac{\partial^2 F}{\partial \boldsymbol{p}\, \partial \boldsymbol{p}} = \frac{\partial^2 \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}}{\partial \boldsymbol{p}\, \partial \boldsymbol{p}} (Z_{\boldsymbol{p}} - Z_{\boldsymbol{k}}) + 2 \frac{\partial \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}}{\partial \boldsymbol{p}} \frac{\partial Z_{\boldsymbol{p}}}{\partial \boldsymbol{p}} + \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \frac{\partial^2 Z_{\boldsymbol{p}}}{\partial \boldsymbol{p}\, \partial \boldsymbol{p}}, \tag{74b}$$

where the terms in $Z_{\boldsymbol{p}} - Z_{\boldsymbol{k}}$ vanish at $\boldsymbol{p} = -\boldsymbol{k}$. It is a straightforward exercise, using the fact that $\theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$ is real, to show that

$$\left(\frac{\partial \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}}{\partial \boldsymbol{p}} \right)\bigg|_{\boldsymbol{p}=-\boldsymbol{k}} = -\frac{1}{2} \frac{\partial \theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}}}{\partial \boldsymbol{k}}. \tag{75}$$

Thus twice the parenthesized term in Eq. (73) can be written in the form

$$2(\cdots) = \theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}} \left(\overline{k^4}(\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) : \frac{\partial^2 Z_{\boldsymbol{k}}}{\partial \boldsymbol{k} \partial \boldsymbol{k}} + 4\overline{k^2} k\, \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right) + \overline{k^4}\hat{\boldsymbol{q}}\cdot \frac{\partial \theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}}}{\partial \boldsymbol{k}} \hat{\boldsymbol{q}}\cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \tag{76a}$$

$$= \hat{\boldsymbol{q}}\cdot \frac{\partial}{\partial \boldsymbol{k}} \left(\overline{k^4}\theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}} \hat{\boldsymbol{q}}\cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right), \tag{76b}$$

so

$$\partial_t Z_{\boldsymbol{k}}^{(\mathrm{A})} = \frac{1}{4} k^2 \sum_{\boldsymbol{q}} s^2 q^2 W_{\boldsymbol{q}} (\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) : \frac{\partial}{\partial \boldsymbol{k}} \left(\overline{k^4}\theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}} \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right) \tag{77a}$$

$$= \frac{1}{4} \sum_{\boldsymbol{q}} s^2 q^2 W_{\boldsymbol{q}} (\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) : \left[\frac{\partial}{\partial \boldsymbol{k}} \left(k^2 \overline{k^4}\theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}} \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right) - 2k\overline{k^4}\theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}} \hat{\boldsymbol{k}} \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right]. \tag{77b}$$

The final step is to move the leftmost $\boldsymbol{k}$ derivative in the first term outside the $\boldsymbol{q}$ sum. That must be done with care because $s = |\hat{\boldsymbol{k}} \times \hat{\boldsymbol{q}}|$ depends on $\boldsymbol{k}$. Now with $c \doteq \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{q}} = \cos\hat{\beta}$ ($\hat{\beta} \doteq \pi - \beta$) and since $\partial\hat{\boldsymbol{k}}/\partial\boldsymbol{k} = (\mathsf{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}})/k$ and $s^2 = 1 - c^2$, one has

$$\frac{\partial c}{\partial \boldsymbol{k}} = \frac{1}{k}(\hat{\boldsymbol{q}} - c\hat{\boldsymbol{k}}), \tag{78a}$$

$$\frac{\partial (s^2)}{\partial \boldsymbol{k}} = -\frac{2c}{k}(\hat{\boldsymbol{q}} - c\hat{\boldsymbol{k}}). \tag{78b}$$

One then readily finds that the contribution from $\partial(s^2)/\partial\boldsymbol{k}$ cancels the second term of Eq. (77b), so

$$\partial_t Z_{\boldsymbol{k}}^{(\mathrm{A})} = \frac{\partial}{\partial \boldsymbol{k}} \cdot \left(\mathsf{D}_{\boldsymbol{k}} \cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right), \tag{79}$$

where $\mathsf{D}_{\boldsymbol{k}}$ is given by Eq. (67a).

Now consider the contributions to the large-$\boldsymbol{k}$ evolution from region B. From Eq. (42b), one has

$$\partial_t Z_{\boldsymbol{k}}^{(\mathrm{B})} = -\frac{1}{2} k^2 \sum_{\boldsymbol{p}\ \mathrm{small}} \left[\left(\frac{p^2}{q^2} \right) s^2 \right] \left(\frac{q^2}{\overline{q^2}} \right) (\overline{q^2} - \overline{p^2})(\overline{q^2} - \overline{k^2}) \times \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} Z_{\boldsymbol{q}} (Z_{\boldsymbol{k}} - Z_{\boldsymbol{p}}), \tag{80}$$

where now $s \doteq \sin\gamma$. It can be verified that the $Z_{\boldsymbol{p}} Z_{\boldsymbol{q}}$ term merely contributes a higher-order correction [of $O(p^4)$] to Eq. (67a). Expansion of the $Z_{\boldsymbol{q}} Z_{\boldsymbol{k}}$ term proceeds generally as before. Note $\overline{q^2} = \overline{k^2} + 2\boldsymbol{k}\cdot\boldsymbol{p} + p^2$, define $F_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \doteq \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} Z_{\boldsymbol{q}}$, introduce the shorthand notation $\overline{\theta}_{\boldsymbol{k}} \equiv \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},-\boldsymbol{k}}$, and expand around $\boldsymbol{q} = -\boldsymbol{k}$:

$$F_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \approx F_{\boldsymbol{k},\boldsymbol{p},-\boldsymbol{k}} - \boldsymbol{p}\cdot \left(\frac{\partial F_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}}{\partial \boldsymbol{q}} \right)\bigg|_{-\boldsymbol{k}} \tag{81a}$$

$$= \overline{\theta}_{\boldsymbol{k}} Z_{\boldsymbol{k}} + \overline{\theta}_{\boldsymbol{k}} \boldsymbol{p}\cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} + \frac{1}{2}\boldsymbol{p}\cdot \frac{\partial \overline{\theta}_{\boldsymbol{k}}}{\partial \boldsymbol{k}} Z_{\boldsymbol{k}}. \tag{81b}$$

Upon collecting terms to dominant order, one has

$$\partial_t Z_{\boldsymbol{k}}^{(\mathrm{B})} = -\frac{1}{2} k^2 Z_{\boldsymbol{k}} \sum_{\boldsymbol{p}\ \mathrm{small}} s^2 p^4 \left(\overline{\theta}_{\boldsymbol{k}} Z_{\boldsymbol{k}} + 2kc\,\overline{\theta}_{\boldsymbol{k}} \hat{\boldsymbol{p}}\cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} + kc\,\hat{\boldsymbol{p}}\cdot \frac{\partial \overline{\theta}_{\boldsymbol{k}}}{\partial \boldsymbol{k}} Z_{\boldsymbol{k}} \right). \tag{82}$$

Further straightforward algebra leads to the result

$$\partial_t Z_{\boldsymbol{k}}^{(\mathrm{B})} = -\partial_{\boldsymbol{k}} \cdot (V_{\boldsymbol{k}} Z_{\boldsymbol{k}}), \tag{83}$$

where (after renaming $\boldsymbol{p} \to \boldsymbol{q}$) $V_{\boldsymbol{k}}$ is defined by Eq. (67b).

2. Evolution of small q's (anisotropic spectrum)

We now apply similar procedures to find the evolution of the small $\boldsymbol{q}$'s, for which, after interchanging $\boldsymbol{k}$ and $\boldsymbol{q}$ in Eq. (42b),

$$\partial_t Z_{\boldsymbol{q}} = -\frac{1}{2} q^2 \sum_{\boldsymbol{k}} s^2 Z_{\boldsymbol{k}} \left(\frac{k^2}{\overline{k^2}} \right) (\overline{p^2} - \overline{k^2})(\overline{q^2} - \overline{k^2}) \times \theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} (Z_{\boldsymbol{q}} - Z_{\boldsymbol{p}}), \tag{84}$$

where $s \doteq \sin \beta$. One must take some care with the domain of integration, which as in Fig. 4 consists of all wave vectors $\boldsymbol{k}$ and $\boldsymbol{p}$ such that either k or p is greater than $k_{\min}$. Referring to Fig. 6, this defines region C ($k \geqslant k_{\min}$) and region D ($k < k_{\min}$, $p \geqslant k_{\min}$). In region C, the angle $\hat{\beta}$ between $\boldsymbol{k}$ and $\boldsymbol{q}$ runs over the entire domain $0 \leqslant \hat{\beta} < 2\pi$, whereas in region D $\hat{\beta}$ is restricted to a domain $-(\pi/2 + \Delta) \leqslant \hat{\beta} \leqslant \pi/2 + \Delta$, where $\Delta \approx q/2k \ll 1$.

Consider first region C. Upon expanding around $\boldsymbol{p} = -\boldsymbol{k}$ and writing $\overline{\theta}_{\boldsymbol{k}} \equiv \theta^{(E)}_{\boldsymbol{q},-\boldsymbol{k},\boldsymbol{k}}$, one finds that through dominant order the contribution to $\gamma_{\boldsymbol{q}} = \frac{1}{2} \partial_t \ln Z_{\boldsymbol{q}}$ is

$$\gamma_{\boldsymbol{q}}^{(\mathrm{C})} = \frac{1}{4} q^4 \sum_{\boldsymbol{k}} s^2 k^2 Z_{\boldsymbol{k}} \left(\overline{\theta}_{\boldsymbol{k}} + \boldsymbol{k} \cdot (\hat{\boldsymbol{q}} \hat{\boldsymbol{q}}) \cdot \frac{\partial \overline{\theta}_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right) \tag{85a}$$

$$= \frac{1}{4} q^4 \sum_{\boldsymbol{k}} Z_{\boldsymbol{k}} \hat{\boldsymbol{q}} \cdot \frac{\partial}{\partial \boldsymbol{k}} (k^3 s^2 c \overline{\theta}_{\boldsymbol{k}}) \tag{85b}$$

$$= -\frac{1}{4} q^4 \sum_{\boldsymbol{k}} k^3 s^2 c \overline{\theta}_{\boldsymbol{k}} \hat{\boldsymbol{q}} \cdot \frac{\partial Z_{\boldsymbol{k}}}{\partial \boldsymbol{k}} + \frac{1}{4} q^4 \frac{1}{N} \left(\oint d\boldsymbol{S}_{\boldsymbol{k}} \cdot \hat{\boldsymbol{q}} k^3 s^2 c \overline{\theta}_{\boldsymbol{k}} Z_{\boldsymbol{k}} \right) \bigg|_{k = k_{\min}}, \tag{85c}$$

where $d\boldsymbol{S}_{\boldsymbol{k}} \cdot \hat{\boldsymbol{q}} = k_{\min} c \, d\beta$. In evaluating formula (84), one can assume to lowest order that $k = k_{\min}$ everywhere except in $p^2 - k^2 \approx 2kq \cos \beta$, and neglect $\Delta \ll 1$:

$$\gamma_{\boldsymbol{q}}^{(\mathrm{D})} \approx -\frac{1}{2} q^3 \sum_{\boldsymbol{p} \in \mathrm{D}} s^2 c (k^3 Z_{\boldsymbol{k}} \theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}})|_{k = k_{\min}}, \tag{86}$$

where

$$\sum_{\boldsymbol{p} \in \mathrm{D}} \approx \frac{1}{N} \int_{-\pi/2}^{\pi/2} d\beta \int_{k_{\min} - q \cos \beta}^{k_{\min}} k \, dk \tag{87a}$$

$$\approx \frac{1}{N} \int_{-\pi/2}^{\pi/2} d\beta \, k_{\min} q \cos \beta. \tag{87b}$$

Because the integrand of Eq. (86) is unchanged under the substitution $\boldsymbol{p} \to -\boldsymbol{p}$, the β integral can be extended to the full domain $[0, 2\pi)$ at the price of a factor of $\frac{1}{2}$. It can then be seen that to lowest order in ϵ the contribution of region D cancels the boundary term in Eq. (85c) and one obtains formula (69a) for $\gamma_{\boldsymbol{q}}$. For more discussion of the significance of region D, see the Appendix.

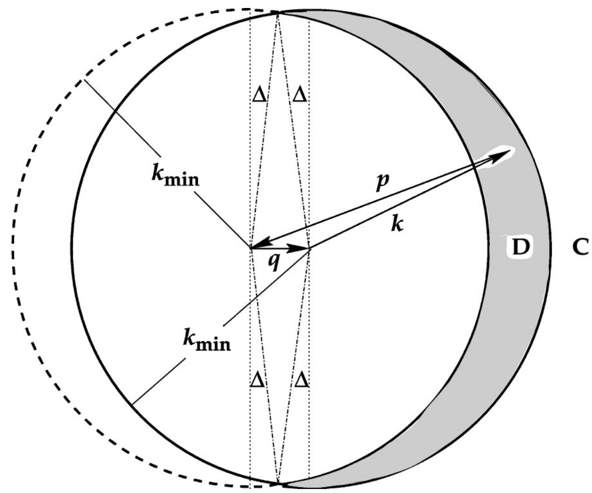

FIG. 6. Integration domains for the evolution of the small $\boldsymbol{q}$'s. Region C: $k \geqslant k_{\min}$. Region D: $k < k_{\min}$, $p \geqslant k_{\min}$. Compare Fig. 4, the isotropic version of this figure.

Finally, the noise contribution can be obtained by symmetrizing formula (84) in $\boldsymbol{k}$ and $\boldsymbol{p}$. (Insert $\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}$ and sum over $\boldsymbol{p}$.) The $Z_{\boldsymbol{q}}$ term merely contributes a higher-order correction to $\gamma_{\boldsymbol{q}}$, while in the $Z_{\boldsymbol{p}}$ term one may set $\boldsymbol{p} = -\boldsymbol{k}$ everywhere except in $p^2 - k^2$. One is immediately led to Eq. (69b).

C. Why $q v^{\mathrm{lin}}_{\mathrm{gr},\boldsymbol{k}}$ does not appear in the denominator of the q-k interaction time

The differences between the triad interaction time $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$ and the response function $\mathcal{R}$ employed in Ref. [2] were enumerated in Sec. I C. With the systematic algebra presented in Secs. III A and III B in hand, we can now discuss why $q v^{\mathrm{lin}}_{\mathrm{gr},\boldsymbol{k}}$ does not, and should not, appear in the denominator of the interaction time, our Eq. (25). The systematic calculations begin with Eq. (16a), which involves the real part of

$$\theta_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{p}} = (\eta^S_{\boldsymbol{q}} + \eta^S_{\boldsymbol{k}} + \eta^S_{\boldsymbol{p}})^{-1}, \tag{88}$$

where $\eta_{\boldsymbol{k}} \doteq i \Omega^{\mathrm{lin}}_{\boldsymbol{k}} + \eta^{\mathrm{nl}}_{\boldsymbol{k}} - \gamma^{\mathrm{lin}}_{\boldsymbol{k}}$. The small-$\boldsymbol{q}$ expansion expands formula (88) around $\boldsymbol{p} = -\boldsymbol{k}$ according to

$$\theta_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{p}} = \theta_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}} - \boldsymbol{q} \cdot \left(\frac{\partial \theta_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{p}}}{\partial \boldsymbol{p}} \right) \bigg|_{-\boldsymbol{k}} + \cdots. \tag{89}$$

However, one could have alternatively performed the expansion of Eq. (88) in the denominator as

$$\theta_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{p}} \approx \left[\eta^S_{\boldsymbol{q}} + \eta^S_{\boldsymbol{k}} + \eta^S_{-\boldsymbol{k}} - \boldsymbol{q} \cdot \left(\frac{\partial \eta^S_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \right) \bigg|_{-\boldsymbol{k}} \right]^{-1}. \tag{90}$$

If the large-$\boldsymbol{k}$ η's are approximated by their linear parts and the linear part of $\eta_{\boldsymbol{q}}$ is ignored, then Eq. (90) reduces to

$$\theta_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{p}} \approx \left[i \left(\mathrm{Im} \, \eta^S_{\boldsymbol{q}} - \boldsymbol{q} \cdot \frac{\partial \Omega^{\mathrm{lin}}_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right) - 2 \gamma^{\mathrm{lin}}_{\boldsymbol{k}} \right]^{-1}, \tag{91}$$

the real part of which is, to within a minus sign, Eq. (26).

We now inquire whether it is valid to proceed from Eq. (90) to Eq. (89) for $q \to 0$. The answer is yes for our fully renormalized form, since as $q \to 0$ the quantity $\eta^S_{\boldsymbol{k}} + \eta^S_{-\boldsymbol{k}} = 2(\mathrm{Re}\,\eta^{\mathrm{nl},S}_{\boldsymbol{k}} - \gamma^{\mathrm{lin}}_{\boldsymbol{k}})$ remains nonzero. Thus the $qv^{\mathrm{lin}}_{\mathrm{gr},\boldsymbol{k}}$ effect is captured by the second and unwritten terms on the right-hand side of Eq. (89). Now in the spectral balance equations only the real part of θ enters, so for the energy balance the effect enters only at $O(q^2)$. However, the calculations in Secs. III A and III B show that only the explicit, $O(q)$ correction in Eq. (89) is relevant to dominant order, and its real part does not contain $qv^{\mathrm{lin}}_{\mathrm{gr},\boldsymbol{k}}$. Thus, although the (real part of the) correction term in Eq. (89) is included in all the formulas involving either wave-number diffusion, drag, or derivatives with respect to $\boldsymbol{k}$ [cf. Eqs. (22) for $\gamma_{\boldsymbol{q}}$], those formulas do not contain any contribution from the $qv^{\mathrm{lin}}_{\mathrm{gr},\boldsymbol{k}}$ term. A consequence is that it is inconsistent to write such formulas using (a renormalized version of) $\mathcal{R}_{\boldsymbol{k};\boldsymbol{q}}$ instead of $\theta_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$; to do so (cf. Refs. [2] and [31]) is to introduce one particular second-order effect into the lowest-order calculation in an uncontrolled way. Note that if there were an actual resonance (if the denominator vanished as $q \to 0$), then the expansion (89) would be invalid; however, that is not the case.

This analysis also explains why, in the heuristic algorithm of Sec. II and in various discussions in Sec. VI, we omit linear physics from the WKE. It does not imply, however, that no effects of wave-packet propagation enter the lowest-order theory. Indeed, we will show in Sec. VI E that the drag term in Eqs. (59) and (66) arises from just that effect. Further discussion is given in Sec. VII.

IV. RENORMALIZED FIELD THEORY FOR DISPARATE INTERACTING SCALES

The direct method of calculation leads one via systematic although possibly obscure algebra to the desired answer. The details are tedious partly because the fundamental formulas (16) treat all scales on equal footing. One might expect that a renormalization procedure that recognizes the presence of disparate scales from the outset would lead to simplified calculations. Accordingly, we present in this section a generalization of the usual renormalization technique that treats the long- and short-wavelength fluctuations as separate components. Actually, in the general version of the calculation one is merely led to the same starting point as before, namely, Eq. (42b), which still requires expansion in ϵ. Nevertheless, it proves useful to introduce the extended formalism, as we will use it in Sec. V to derive an alternative method of calculation that will motivate the heuristic energy algorithm presented in Sec. II.

There is one important caveat to the calculations to follow, relating to the issues of random Galilean invariance discussed in Sec. I B. The variational procedure to be employed is firmly rooted in Eulerian correlation functions, so it is not naturally random-Galilean-invariant. This means that, if one were to work out the nonlinear renormalizations explicitly, one would in all formulas obtain $\eta_{\boldsymbol{k}}$, not $\eta^S_{\boldsymbol{k}}$. This is not a critical issue, since from the calculations in Sec. III one already knows the correct form of the answer and can insert η^S's by hand if necessary. (Actually, we merely couch the results in terms of $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$.) However, it is an important deficiency of the general method. (Lagrangian or mixed Eulerian-Lagrangian techniques [33] are useful in this context, but are beyond the scope of this article.)

For notational simplicity, we shall illustrate the procedure for pure HM dynamics. However, the techniques and general form of the results apply to generalized HM dynamics as well.

A. Background: The Martin-Siggia-Rose approach to classical field theory

In seminal work, Martin, Siggia, and Rose (MSR) [34] showed how to adapt powerful methods of quantum field theory to the problem of classical statistical dynamics. The basic idea is to construct generating functionals for correlation and response functions in terms of a classical action built from the given nonlinear dynamics in the presence of a vector of external sources $\boldsymbol{\eta}$. (There should be no confusion between this $\boldsymbol{\eta}$ and the renormalized damping $\eta_{\boldsymbol{k}}$ in the Markovian closures.) Generalized n-point correlation functions follow from n derivatives of the generating functional with respect to appropriate sources; physical correlations are obtained in the limit $\boldsymbol{\eta} \to 0$. Repeated differentiation with respect to $\boldsymbol{\eta}$ merely generates an unclosed multipoint statistical hierarchy (a multitime generalization of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy of classical kinetic theory). However, by making a Legendre transformation [35,36] from $\boldsymbol{\eta}$ to the mean field $\langle \varphi \rangle_{\boldsymbol{\eta}} \equiv \langle \varphi \rangle[\boldsymbol{\eta}]$ (square brackets denote functional dependence), one can define vertex functions, the (functional) equations for which can be usefully truncated to obtain approximate and closed renormalized equations. The DIA emerges by neglecting vertex renormalization altogether; the second-order Markovian closures have a similar interpretation [37]. The MSR technique was reviewed by Krommes [19], who provided many references to the original papers; see also the introductory discussion in Chap. 9 of Ref. [38]. Probably the most useful realization of the formalism employs path-integral techniques, as discussed by Jensen [39] and in the references therein.

Thus, to effect MSR renormalization of the pure HM equation, one supplements the original dynamical equation (6) with a scalar, statistically sharp source $\hat{\eta}$:

$$(\alpha - \nabla^2)\partial_t \varphi(\boldsymbol{x},t) + V_* \partial_y \varphi + \boldsymbol{V}_E \cdot \boldsymbol{\nabla}(-\nabla^2 \varphi) = \hat{\eta}(\boldsymbol{x},t). \tag{92}$$

(We write ∇^2 instead of $\nabla^2_\perp$, and also write α instead of $\hat{\alpha}$ to avoid confusion with other uses of the caret in this section.) Such a source breaks the statistical symmetry and induces a nonzero mean field $\langle \varphi \rangle_{\hat{\eta}}$ functionally dependent on $\hat{\eta}$. More generally, a conjugate field $\hat{\varphi}$ is also introduced that obeys an adjoint dynamics (not written here, but coupled to those of φ) and interacts with an external source η. Averages with respect to the enlarged ensemble that embraces both φ and $\hat{\varphi}$ lead to mean fields and higher-order cumulants (denoted by $\langle\langle \cdots \rangle\rangle$) that depend on both η and $\hat{\eta}$. We write $(\varphi,\hat{\varphi})^T \equiv \boldsymbol{\varphi}$ and $(\eta,\hat{\eta})^T \equiv \boldsymbol{\eta}$ (T denotes transpose) and must consider the mean field $\langle\langle \boldsymbol{\varphi} \rangle\rangle_{\boldsymbol{\eta}}$. It can be shown that cumulants of the physical field φ follow by functional differentiation with respect to η, and that mixed derivatives with respect to both η

and $\hat{\eta}$ generate various infinitesimal response functions. For example, with $1\equiv(\boldsymbol{x}_1,t_1)$ and $C_1(1)\equiv\langle\langle\varphi\rangle\rangle(1)=\langle\varphi\rangle(1)$, one has

$$C_2(1,2)\equiv\langle\langle\varphi(1)\varphi(2)\rangle\rangle=\langle\delta\varphi(1)\delta\varphi(2)\rangle=\frac{\delta C_1(1)}{\delta\eta(2)} \tag{93}$$

and, in general,

$$C_{n+1}(1,2,\ldots,n,n+1)=\frac{\delta C_n(1,2,\ldots,n)}{\delta\eta(n+1)}. \tag{94}$$

The fundamental two-point infinitesimal response function $R(1;1')$ is

$$R(1;1')=\delta C_1(1)/\delta\hat{\eta}(1'). \tag{95}$$

To avoid clutter, we do not explicitly indicate the dependence of the cumulants on $\boldsymbol{\eta}$, or that physical observables obtain their values at $\boldsymbol{\eta}=\mathbf{0}$. However, in generating various statistical equations, a key caveat is that one must not set mean fields to zero until the very end of the calculation, because they are nonvanishing in the perturbed ensemble and may be differentiated. Also note that the functional derivatives employed here are parametrized by both time and space, whereas the operator employed in Sec. II (and in Ref. [2]) is parametrized only by space.

For quadratically nonlinear dynamics of the form

$$\partial_t\varphi+iL\varphi=O(\varphi\varphi), \tag{96}$$

where L is a linear operator, these considerations lead to the general set of coupled Dyson equations

$$\partial_t R(1;1')+iLR+\int d\overline{1}\,\Sigma^{\rm nl}(1;\overline{1})R(\overline{1};1')=\delta(1-1'), \tag{97a}$$

$$\partial_t C(1,1')+iLC+\int d\overline{1}\,\Sigma^{\rm nl}(1;\overline{1})C(\overline{1},1')=\int d\overline{1}F^{\rm nl}(1,\overline{1})R(1';\overline{1}), \tag{97b}$$

where $\Sigma^{\rm nl}$ and $F^{\rm nl}$ represent the set of all connected diagrams that arise from the nonlinearity. $\Sigma^{\rm nl}$ is a generalized nonlinear damping, and $F^{\rm nl}$ can be interpreted as the covariance of an internal ‘‘incoherent’’ noise arising from the nonlinear mode coupling [40]. [The semicolon notation is used here to indicate functions that are one sided (causal) in time; thus $R(t;t')\propto H(t-t')$, where H is the unit step function. More generally, the coordinate(s) after the semicolon can also be interpreted as signifying the point(s) in space and time at which a perturbation is applied, and we will use that notation later in considering responses to perturbing $\varphi_{\boldsymbol{q}}$'s.] In various closures such as the DIA, explicit forms are obtained for $\Sigma^{\rm nl}$ and $F^{\rm nl}$. For present purposes, we just note that, when the dynamical equation is written in the conservation form

$$\partial_t\varphi+iL\varphi+\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{\Gamma}}=0, \tag{98}$$

one finds the general formulas

$$\Sigma^{\rm nl}(1;\overline{1})=\frac{\delta\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}(1)}{\delta P(\overline{1})}, \tag{99a}$$

$$F^{\rm nl}(1,\overline{1})=-\frac{\delta\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}(1)}{\delta\hat{P}(\overline{1})}, \tag{99b}$$

where $\boldsymbol{\Gamma}\doteq\langle\tilde{\boldsymbol{\Gamma}}\rangle$. These results are well known to workers in statistical field theory [34,19].

B. Wave-number filtering and the long-wavelength growth rate

If one were merely to proceed in this vein, one would arrive at the DIA by neglecting vertex renormalization. The content and implications of the DIA are very well understood [19,21], so need not be discussed here. The further reduction of the DIA to Markovian form was discussed at length in Ref. [37] for the general case of weakly inhomogeneous and nonstationary statistics; for statistically homogeneous situations, one arrives at Eqs. (16). Additional discussion of Markovian approximations was given in Ref. [20]. We are interested here in developing a variant of the renormalization procedure that treats the long- and short-wavelength fluctuations as separate components. Accordingly, we decompose the potential into a short-wavelength component $\varphi^>$ and a long-wavelength component $\varphi^<$: $\varphi=\varphi^>+\varphi^<$. For example, a realization of $\varphi^>$ is $\varphi^>(\boldsymbol{x})=(2\pi)^{-d}\int_{k\geq k_\star}d\boldsymbol{k}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\varphi(\boldsymbol{k})$ and similarly $\varphi^<=\int_{k<k_\star}\cdots$, where $k_\star$ is some wave number intermediate in the spectrum. Note that, if one wants to strictly match this formalism to the expansions in ϵ developed in Sec. III, one should actually introduce three filtered fields: $\varphi^>=\int_{k\geq k_{\rm min}}\cdots$, $\varphi^<=\int_{q\leq q_{\rm max}}\cdots$, and a field with intermediate wave-number content $\varphi^{(\rm mid)}$. We shall not introduce $\varphi^{(\rm mid)}$ explicitly, relying instead on subsidiary approximations to obtain the disparate-scale interactions between $\varphi^>$ and $\varphi^<$.

Such wave-number filtering is a projection operation. Therefore, upon projecting Eq. (92) (in the presence of an external source $\hat{\eta}$) onto the $>$ and $<$ subspaces, one obtains

$$\overset{(1)}{(\alpha-\nabla^2)\partial_t\varphi^>}+\overset{(2)}{V_*\partial_y\varphi^>}+\overset{(3)}{[\boldsymbol{V}_E^>\cdot\boldsymbol{\nabla}(-\nabla^2\varphi^>)]^>}+\overset{(4)}{\boldsymbol{V}_E^>\cdot\boldsymbol{\nabla}(-\nabla^2\varphi^<)}+\overset{(5)}{\boldsymbol{V}_E^<\cdot\boldsymbol{\nabla}(-\nabla^2\varphi^>)}\approx(\alpha-\nabla^2)\hat{\eta}^>, \tag{100a}$$

$$\overset{(1')}{(\alpha-\nabla^2)\partial_t\varphi^<}+\overset{(2')}{V_*\partial_y\varphi^<}+\overset{(3')}{\boldsymbol{V}_E^<\cdot\boldsymbol{\nabla}(-\nabla^2\varphi^<)}+\overset{(4')}{[\boldsymbol{V}_E^>\cdot\boldsymbol{\nabla}(-\nabla^2\varphi^>)]^<}=(\alpha-\nabla^2)\hat{\eta}^<. \tag{100b}$$

[In writing Eq. (100a), we used the fact that the self-interactions between $\varphi^<$ cannot contribute to $\varphi^>$ except near the boundary between $\varphi^>$ and $\varphi^<$ in wave-number space. We ignore that interaction; it could be taken into account by the introduction of $\varphi^{(\mathrm{mid})}$.] The various terms are numbered to help one follow the algebra. [We eschew the more meaningful numbering scheme $(1^>)$ and $(1^<)$ as being too cumbersome.]

Our goal is to study statistics in a spatially homogeneous background. To ensure that, one must ultimately average over the statistics of both $\varphi^>$ and $\varphi^<$. As usual, we shall denote the full statistical average by $\langle\cdots\rangle$. Later we shall discuss a conditional averaging procedure wherein one averages only over the short-wavelength statistics. However, here we shall employ unconditional statistical averaging to derive a general formula for the long-wavelength growth rate. The cumulant average of Eq. (100b) is

$$\overset{(1')}{(\alpha-\nabla^2)\partial_t\langle\langle\varphi^<\rangle\rangle}+\overset{(2')}{V_*\partial_y\langle\langle\varphi^<\rangle\rangle}+\overset{(3')}{\langle\langle \boldsymbol{V}_E^<\rangle\rangle\cdot\boldsymbol{\nabla}(-\nabla^2\langle\langle\varphi^<\rangle\rangle)}$$
$$+\overset{(4')}{\boldsymbol{\nabla}\cdot\langle\langle\boldsymbol{V}_E^<(-\nabla^2\varphi^<)\rangle\rangle}+\overset{(5')}{[\langle\langle\boldsymbol{V}_E^>\rangle\rangle\cdot\boldsymbol{\nabla}(-\nabla^2\langle\langle\varphi^>\rangle\rangle)]^<}$$
$$+\overset{(6')}{\boldsymbol{\nabla}\cdot\langle\langle\boldsymbol{V}_E^>(-\nabla^2\varphi^>)\rangle\rangle^<}$$
$$=(\alpha-\nabla^2)\hat{\eta}^<. \tag{101}$$

According to the MSR procedure, one obtains the equation for the long-wavelength covariance $C^<\doteq\langle\langle\varphi^<\varphi^<\rangle\rangle\equiv C^{<<}$ by functionally differentiating Eq. (101) with respect to $\eta^<$. If one writes $V_E(1)=\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})\varphi(\overline{1})$ [where $\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})=\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\,\delta(1-\overline{1})$ or $\hat{\boldsymbol{\mathcal{V}}}_{\boldsymbol{k}}(t,\overline{t})=\hat{\boldsymbol{z}}\times i\boldsymbol{k}\delta(t-\overline{t})$], then one obtains

$$\overset{(1')}{(\alpha-\nabla^2)\partial_t C^<(1,1')}+\overset{(2')}{V_*\partial_y C^<(1,1')}+\overset{(3'\mathrm{a})}{\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})C^<(\overline{1},1')\cdot\boldsymbol{\nabla}(-\nabla^2\underline{\langle\langle\varphi^<\rangle\rangle})}+\overset{(3'\mathrm{b})}{\underline{\langle\langle\boldsymbol{V}_E^<\rangle\rangle}\cdot\boldsymbol{\nabla}[-\nabla^2C^<(1,1')}$$
$$+\overset{(4')}{\delta\langle\langle\mathrm{self}^<\rangle\rangle/\delta\eta(1')}+\overset{(5'\mathrm{a})}{[\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})C^{><}(1,1')\cdot\boldsymbol{\nabla}(-\nabla^2\underline{\langle\langle\varphi^>\rangle\rangle})]^<}+\overset{(5'\mathrm{b})}{\{\underline{\langle\langle\boldsymbol{V}_E^>\rangle\rangle}\cdot\boldsymbol{\nabla}[-\nabla^2C^{><}(1,1')]\}^<}$$
$$+\overset{(6')}{\delta\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}^<(1)/\delta\eta(1')}=0, \tag{102}$$

where $\boldsymbol{\Gamma}\doteq\langle\langle\boldsymbol{V}_E^>(-\nabla^2\varphi^>)\rangle\rangle$ is the flux of vorticity. In the limit $\boldsymbol{\eta}\to\boldsymbol{0}$, the underlined terms vanish for homogeneous statistics. The self-interaction term $(4')$ generates a nonlinear damping rate that renormalizes the linear long-wavelength propagator. We shall not work out that effect explicitly for two reasons: first, our goals here are merely to show consistency with the systematic, fully renormalized calculations we have already done in Sec. III and to identify the key terms that contribute under long-wavelength modulation to the final answer; second, the resulting Eulerian renormalizations would not be random Galilean invariant. However, upon Legendre transforming from η to $\langle\langle\varphi\rangle\rangle_\eta$, the derivative in term $(6')$ of Eq. (102) becomes

$$\frac{\delta}{\delta\eta^<(1')}=\underbrace{\overset{(6'\mathrm{a})}{C^<(\bar{1},1')\frac{\delta}{\delta P^<(\bar{1})}}}_{\text{long-wavelength damping}}+\underbrace{\overset{(6'\mathrm{b})}{R^<(1';\bar{1})\frac{\delta}{\delta\hat{P}^<(\bar{1})}}}_{\text{long-wavelength noise}}+\overset{(6'\mathrm{c})}{C^{><}(\bar{1},1')\frac{\delta}{\delta P^>(\bar{1})}}+\overset{(6'\mathrm{d})}{R^{<>}(1';\bar{1})\frac{\delta}{\delta\hat{P}^>(\bar{1})}}. \tag{103}$$

In the limit $\boldsymbol{\eta}\to0$, cross correlations between long- and short-wavelength fluctuations vanish to lowest order, so we ignore the last two terms in Eq. (103). Upon passing to the physical limit $\boldsymbol{\eta}\to\boldsymbol{0}$, one then obtains

$$\overset{(1')}{(\alpha-\nabla^2)\partial_t C^<(1,1')}+\overset{(2')}{V_*\partial_y C^<(1,1')}$$
$$+\overset{(4)}{(\text{renormalized damping})^<}$$
$$+\overset{(6'\mathrm{a})}{(\alpha-\nabla^2)\Sigma^{\mathrm{nl},<}(1,\overline{1})C^<(\overline{1},1')}=\overset{(6'\mathrm{b})}{(\text{noise})^<}, \tag{104}$$

where

$$\Sigma^{\mathrm{nl},<}(1,\overline{1})\doteq(\alpha-\nabla^2)^{-1}\frac{\delta\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}^<(1)}{\delta P^<(\overline{1})}. \tag{105}$$

The damping rate $\Sigma^{\mathrm{nl},<}$ is closely related to the desired $\gamma_{\boldsymbol{q}}$. To reduce Eq. (105) further to a Markovian form, one must introduce the fluctuation-dissipation ansatz

$$C_{\boldsymbol{q}}(t,\overline{t})=R_{\boldsymbol{q}}(t;\overline{t})C_{\boldsymbol{q}}(t)\quad(t\geqslant\overline{t}) \tag{106}$$

(see discussions in Refs. [37] and [20]) and consider Eq. (104) for equal times $t'=t$. With $C(t,t)\equiv C(t)$, one therefore finds

$$\partial_tC_{\boldsymbol{q}}(t)-2\gamma_{\boldsymbol{q}}C_{\boldsymbol{q}}(t)=\ \ldots, \tag{107}$$

where

$$\gamma_{\boldsymbol{q}} \doteq -\mathrm{Re}\int_{-\infty}^{t} d\overline{t}\, \Sigma_{\boldsymbol{q}}^{\mathrm{nl},<}(t;\overline{t}) R_{\boldsymbol{q}}^{<}(t;\overline{t})^{*} \tag{108a}$$

$$= -\left(\frac{1}{\alpha_{\boldsymbol{q}}+q^2}\right)\mathrm{Re}\int_{-\infty}^{t} dt' \frac{\delta(\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}_{\boldsymbol{q}})(t)}{\delta P_{\boldsymbol{q}}(t')} R_{\boldsymbol{q}}^{*}(t;t'). \tag{108b}$$

We have dropped the $<$ superscript on $\boldsymbol{\Gamma}_{\boldsymbol{q}}$, $P_{\boldsymbol{q}}$, and $R_{\boldsymbol{q}}$ because the $\boldsymbol{q}$ subscript already denotes the long-wavelength projection. The factor of $R_{\boldsymbol{q}}$ in Eq. (108b) is the formalism's way of introducing the long-wavelength autocorrelation time $\sim\eta_{\boldsymbol{q}}^{-1}$, which is required because the short-wavelength modes $\boldsymbol{k}$ and $\boldsymbol{p}$ are interacting nonlinearly with mode $\boldsymbol{q}$. Ultimately, it will be seen that the presence of $R_{\boldsymbol{q}}$ is essential in order to recover expressions involving the correct triad interaction time $\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$. (There is no significant distinction between $R_{\boldsymbol{q}}$ and $R_{\boldsymbol{q}}^{S}$ because small-$\boldsymbol{q}$ modulations cannot be advected by substantially longer wavelengths.) However, one can obtain an approximate formula by dropping the time integral and $R_{\boldsymbol{q}}$, provided that one then parametrizes the functional derivative only by space, not time, and inserts the appropriate θ by hand. The resulting formula

$$\gamma_{\boldsymbol{q}} = -\left(\frac{1}{\alpha_{\boldsymbol{q}}+q^2}\right)\mathrm{Re}\frac{\delta(\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}_{\boldsymbol{q}})}{\delta P_{\boldsymbol{q}}} \tag{109}$$

is essentially the one employed in Ref. [2], except that those authors differentiated with respect to the random potential $\tilde{\varphi}_{\boldsymbol{q}}$. One can show that formulas (108b) and (109) hold for generalized HM dynamics as well.

To simplify Eq. (108b) further, we observe that $\boldsymbol{\Gamma}(1)$ can be obtained from certain differential operations applied to $C(1,2)$, after which the limit $2\rightarrow 1$ may be taken:

$$\boldsymbol{\Gamma}(1) = \langle\langle \boldsymbol{V}_E(1)\cdot\boldsymbol{\nabla}_1[-\nabla_1^2\varphi(1)]\rangle\rangle$$
$$= \hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})\cdot\boldsymbol{\nabla}_2(-\nabla_2^2)C(\overline{1},2)\big|_{2=1}.$$

The functional differentiation required in Eq. (105) introduces the function

$$\hat{C}(1,2;1') \doteq \frac{\delta C^{>}(1,2)}{\delta P(1')}. \tag{110}$$

Note that by homogeneity $\hat{C}(1,2;1')$ must be a function of just two spatial difference variables. We shall adopt the convention (temporarily ignoring the time arguments and denoting spatial variables by underlines) $\hat{C}(\underline{1},\underline{2};\underline{1}') = \hat{C}(\underline{1}-\underline{2};\underline{1}-\underline{1}')$. Upon using $\boldsymbol{k}$ as a wave vector for the $>$ coordinates 1 and 2 and using $\boldsymbol{q}$ for the $<$ coordinate $1'$, one has

$$\hat{C}(1,2;1') = \sum_{\boldsymbol{k},\boldsymbol{q}} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)} e^{i\boldsymbol{q}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_1')}\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t_1,t_2;t'). \tag{111}$$

Upon inserting this into Eq. (108b), one is finally led to the general Markovian formula

$$\gamma_{\boldsymbol{q}} = (\alpha_{\boldsymbol{q}}+q^2)^{-1}\mathrm{Re}\sum_{\boldsymbol{k}} \hat{\boldsymbol{z}}\cdot(\boldsymbol{q}\times\boldsymbol{k})k^2 \times\int_{-\infty}^{t} dt'\, \hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t,t;t')R_{\boldsymbol{q}}^{*}(t;t'). \tag{112}$$

An equation for $\hat{C}$ follows by successive differentiations of Eq. (100a) with respect to $\eta^{>}$ and P. Upon differentiating with respect to $\eta^{>}$, one obtains

$$0 = (\alpha-\nabla^2)\overset{(1)}{\partial_t C^{>}(1,2)} + \overset{(2)}{V_{*}\partial_y C^{>}(1,2)} + \frac{\delta}{\delta\eta^{>}(2)}\overset{(3)}{(\text{self terms})^{>}(1)} + \overset{(4a)}{\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})C^{>}(\overline{1},2)\cdot\boldsymbol{\nabla}[-\nabla^2\underline{P(1)}]}$$
$$+\overset{(4b)}{\underline{\langle\langle \boldsymbol{V}_E^{>}\rangle\rangle(1)}\cdot\boldsymbol{\nabla}[-\nabla^2 C^{<>}(1,2)} + \frac{\delta}{\delta\eta^{>}(2)}\overset{(4c)}{\boldsymbol{\nabla}\cdot\langle\langle \boldsymbol{V}_E^{>}(-\nabla^2\varphi^{<})\rangle\rangle} + \overset{(5a)}{\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})C^{<>}(\overline{1},2)\cdot\boldsymbol{\nabla}[-\nabla^2\underline{P^{>}(1)}]}$$
$$+\overset{(5b)}{\underline{\langle\langle \boldsymbol{V}_E^{<}\rangle\rangle}\cdot\boldsymbol{\nabla}[-\nabla^2 C^{>}(1,2)} + \frac{\delta}{\delta\eta^{>}(2)}\overset{(5c)}{\boldsymbol{\nabla}\cdot\langle\langle \boldsymbol{V}_E^{<}(-\nabla^2\varphi^{>})\rangle\rangle}, \tag{113}$$

where the mean fields have again been underlined. Upon differentiating this result with respect to $P(1')$, one finds

$$0 = (\alpha-\nabla^2)\overset{(1)}{\partial_t\hat{C}(1,2;1')} + \overset{(2)}{V_{*}\partial_y\hat{C}(1,2;1')} + \overset{(3)}{(\text{self terms})} + \overset{(4a)}{\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})\hat{C}(\overline{1},2;1')\cdot\boldsymbol{\nabla}[-\nabla^2\underline{P(1)}]}$$
$$+\overset{(4a)}{\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})C^{>}(\overline{1},2)\cdot\boldsymbol{\nabla}[-\nabla^2\delta(1-1')]} + \overset{(4b)}{\frac{\delta\langle\langle \boldsymbol{V}_E^{>}(1)\rangle\rangle}{\delta P(1')}\cdot\boldsymbol{\nabla}[-\nabla^2 C^{<>}(1,2)} + \underline{\langle\langle \boldsymbol{V}_E^{>}\rangle\rangle}\cdot\boldsymbol{\nabla}\left(\overset{(4b)}{-\nabla^2}\frac{\delta C^{<>}(1,2)}{\delta P(1')}\right)$$
$$+[\text{term (4c)}] + \overset{(5a)}{\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})\frac{\delta C^{<>}(\overline{1},2)}{\delta P(1')}\cdot\boldsymbol{\nabla}[-\nabla^2\underline{P^{>}(1)}]} + \overset{(5a)}{\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})C^{<>}(\overline{1},2)\cdot\boldsymbol{\nabla}\left(-\nabla^2\frac{\delta P^{>}(1)}{\delta P(1')}\right)}$$
$$+\overset{(5b)}{\hat{\boldsymbol{\mathcal{V}}}(1,1')\cdot\boldsymbol{\nabla}[-\nabla^2 C^{>}(1,2)} + \overset{(5b)}{\underline{\boldsymbol{V}_E^{<}(1)}\cdot\boldsymbol{\nabla}[-\nabla^2\hat{C}(1,2;1')]} + [\text{term (5c)}]. \tag{114}$$

Later we shall differentiate this equation yet again, so keeping track of the mean fields is important. However, for present purposes one can obtain the equation for the physical $\hat{C}$ by setting the mean fields and cross correlations to zero. Thus, upon moving terms not involving $\hat{C}$ to the right-hand side and indicating nonlinear renormalizations by $\cdots$, one obtains

$$(\alpha-\nabla_1^2)\partial_{t_1}\hat{C}(1,2;1')+V_*\partial_{y_1}\hat{C}(1,2;1')+\cdots$$
$$=-\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})C^{>}(\overline{1},2)\cdot\boldsymbol{\nabla}[-\nabla^2\delta(1-1')]$$
$$-\hat{\boldsymbol{\mathcal{V}}}(1,1')\cdot\boldsymbol{\nabla}[-\nabla^2 C^{>}(1,2)]. \tag{115}$$

The spatial Fourier transform of Eq. (115) is

$$\partial_t\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t_1,t_2;t')+i\Omega_{-\boldsymbol{p}}^{\text{lin}}\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t_1,t_2;t')+\cdots$$
$$=\hat{\boldsymbol{z}}\cdot(\boldsymbol{k}\times\boldsymbol{q})A_{\boldsymbol{k};\boldsymbol{q}}(t_1,t_2)\,\delta(t_1-t'), \tag{116}$$

where $\boldsymbol{p}\doteq-\boldsymbol{k}-\boldsymbol{q}$,

$$A_{\boldsymbol{k};\boldsymbol{q}}(t_1,t_2)\doteq(\alpha_{\boldsymbol{p}}+p^2)^{-1}(q^2-k^2)C_{\boldsymbol{k}}^{>}(t_1,t_2), \tag{117}$$

and $\Omega_{\boldsymbol{k}}^{\text{lin}}$ is the linear diamagnetic frequency defined by Eq. (7a). For Eq. (112), one requires $\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t,t;t')$. Now $\partial_t\hat{C}(\underline{1},t,\underline{2},t;1')=[\partial_{t_1}\hat{C}(1,2;1')+\partial_{t_2}\hat{C}(2,1;1')]|_{t_1=t_2=t}$, and from Eq. (111) one has $\hat{C}(2,1;1')_{\boldsymbol{k};\boldsymbol{q}}=\hat{C}_{\boldsymbol{p};\boldsymbol{q}}(t_2,t_1;t')$. Thus one obtains

$$\partial_t\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t,t;t')-i\Omega_{\boldsymbol{p}}^{\text{lin}}\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t,t;t')-i\Omega_{\boldsymbol{k}}^{\text{lin}}\hat{C}_{\boldsymbol{p};\boldsymbol{q}}(t,t;t')+\cdots$$
$$=S_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}(t)\,\delta(t-t'), \tag{118}$$

where

$$S_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}(t)\doteq\hat{\boldsymbol{z}}\cdot(\boldsymbol{k}\times\boldsymbol{q})[A_{\boldsymbol{k};\boldsymbol{q}}(t,t)-A_{\boldsymbol{p};\boldsymbol{q}}(t,t)] \tag{119}$$

is symmetric in $\boldsymbol{k}$ and $\boldsymbol{p}$. From Eq. (118) follows an equation for the desired quantity $\mathcal{C}_{\boldsymbol{k};\boldsymbol{q}}(t;t')\doteq\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t,t;t')R_{\boldsymbol{q}}^*(t;t')$:

$$\partial_t\mathcal{C}_{\boldsymbol{k};\boldsymbol{q}}(t;t')-i(\Omega_{\boldsymbol{k}}^{\text{lin}}+\Omega_{\boldsymbol{p}}^{\text{lin}}+\Omega_{\boldsymbol{q}}^{\text{lin}})\mathcal{C}_{\boldsymbol{k};\boldsymbol{q}}(t;t')+\cdots$$
$$+i\Omega_{\boldsymbol{k}}^{\text{lin}}[\mathcal{C}_{\boldsymbol{k};\boldsymbol{q}}(t;t')-\mathcal{C}_{\boldsymbol{p};\boldsymbol{q}}(t;t')]=0\quad(t>t'), \tag{120}$$

with initial condition $\mathcal{C}_{\boldsymbol{k};\boldsymbol{q}}(t';t')=S_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}(t)$. Were it not for the $\mathcal{C}_{\boldsymbol{k};\boldsymbol{q}}-\mathcal{C}_{\boldsymbol{p};\boldsymbol{q}}\equiv\Delta\mathcal{C}_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}$ term, the operator on the left-hand side of Eq. (120) would be identical to the complex conjugate of the one that defines the inverse of the triad interaction time. However, we will show that the contribution of that term to the final answer vanishes. To do so, we integrate Eq. (120) over $\tau\doteq t-t'$ and define $\overline{\mathcal{C}}\doteq\int_0^\infty d\tau\mathcal{C}(\tau)$. One can verify that $\mathcal{C}(\infty)=0$. Therefore

$$(\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}^*)^{-1}\overline{\mathcal{C}}_{\boldsymbol{k};\boldsymbol{q}}+i\Omega_{\boldsymbol{k}}^{\text{lin}}\Delta\overline{\mathcal{C}}_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}=S_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}. \tag{121}$$

From Eq. (121), form an equation for $\Delta\overline{\mathcal{C}}$ by interchanging $\boldsymbol{k}$ and $\boldsymbol{p}$ and subtracting the resulting equations. The right-hand side vanishes because S is symmetric, so one obtains

$$(\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}^*)^{-1}\Delta\overline{\mathcal{C}}_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}+i(\Omega_{\boldsymbol{k}}^{\text{lin}}-\Omega_{\boldsymbol{p}}^{\text{lin}})\Delta\overline{\mathcal{C}}_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}=0. \tag{122}$$

The unique solution of this homogeneous linear equation is $\Delta\overline{\mathcal{C}}_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}=0$. Therefore the solution of Eq. (121) is

$$\overline{\mathcal{C}}_{\boldsymbol{k};\boldsymbol{q}}=\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}^*S_{\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}}. \tag{123}$$

Upon inserting this into Eq. (112), one finds

$$\gamma_{\boldsymbol{q}}=-(\alpha_{\boldsymbol{q}}+q^2)^{-1}\text{Re}\sum_{\boldsymbol{k}}(\hat{\boldsymbol{z}}\cdot\boldsymbol{q}\times\boldsymbol{k})^2k^2\,\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}(A_{\boldsymbol{k};\boldsymbol{q}}-A_{\boldsymbol{p};\boldsymbol{q}}) \tag{124a}$$

$$=(\alpha_{\boldsymbol{q}}+q^2)^{-1}\sum_{\boldsymbol{k},\boldsymbol{p}\text{ large}}\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}(\hat{\boldsymbol{z}}\cdot\boldsymbol{q}\times\boldsymbol{k})^2$$
$$\times(p^2-k^2)\,\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}^r A_{\boldsymbol{k};\boldsymbol{q}}. \tag{124b}$$

Upon recalling the definitions (43) of $\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}^{(E)}$, (117) of $A_{\boldsymbol{k};\boldsymbol{q}}$, and (13) of $W_{\boldsymbol{k}}$, one readily finds

$$\gamma_{\boldsymbol{q}}=\frac{1}{4}q^2\sum_{\boldsymbol{k},\boldsymbol{p}\text{ large}}\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}\sin^2\beta(p^2-k^2)(q^2-k^2)\,\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}^{(E)}W_{\boldsymbol{k}}. \tag{125}$$

This result is identical to the formula for $\gamma_{\boldsymbol{q}}$ that follows from Eq. (84) by dropping the bars and retaining only the $W_{\boldsymbol{q}}$ term.

This calculation is merely an extensive consistency check. It shows the equivalence between (i) the direct method of renormalizing all interactions at once, then extracting the contributions due to interactions between disparate scales; and (ii) an initial filtering into short- and long-wavelength components, followed by renormalization of this extended multicomponent system. (We did not perform all renormalizations in detail here, being content to display the flow of the logic and to identify the particular terms that contribute to the final answer.) In its raw form, it does not provide algebraic simplifications over the direct method. However, it does illustrate the variational procedure for obtaining statistical equations. We shall now use that procedure to give an alternate derivation of formula (125) based on a nonlinear statistical Poynting theorem.

V. THE LONG-WAVELENGTH GROWTH RATE FROM RIGOROUS ENERGETICS

The authors of Ref. [2] advocated a method of calculation based on a quasilinear wave-energy or Poynting-theorem approach. As we discussed in Sec. I C, their results are not in complete agreement with ours, either in mathematical detail or, more importantly, in physical interpretation, so it is useful to examine their procedure in detail. We shall first proceed systematically, using the techniques of renormalized field theory introduced in the last section, and show how one can recover the correct long-wavelength growth rate $\gamma_{\boldsymbol{q}}$ [Eq. (69a)] to dominant order in ϵ. (With extra work, one can also recover the noise terms, but we omit those details as they do not add additional insight.) Subsequent discussion and heuristic examples will demonstrate why interpretations and algorithms based on linear physics fail.

A. A nonlinear statistical Poynting theorem

We shall derive a nonlinear energy-conservation theorem. The use of such theorems is well established in statistical plasma theory. Dupree and Tetreault [41] discussed deficiencies of standard resonance-broadening theory with respect to energy conservation. Similon [42] showed in detail why the DIA was properly energy conserving and why various simpler closures such as resonance-broadening theory were not. Krommes [19] reviewed the general procedure of obtaining statistical energy-conservation theorems in the context of unmagnetized Vlasov dynamics. The fundamental operation is to multiply the continuity equation for fluctuating charge density by the fluctuating potential, then average over space and integrate by parts, thereby obtaining an expression for the time rate of change of the mean fluctuation energy. Now if such a procedure is applied to the basic HM equation in the absence of linear growth or dissipation, a trivial result is obtained: one merely finds that total energy does not evolve at all under the action of either the linear or nonlinear terms. Put another way, fluctuation energy evolves because of Ohmic heating $\langle \delta\boldsymbol{j}\cdot\delta\boldsymbol{E}\rangle$. For perpendicular currents comprising polarization and gyrocenter contributions, the $\langle \delta\boldsymbol{j}_\perp\cdot\delta\boldsymbol{E}_\perp\rangle$ term vanishes and only the dissipative parts of $\langle \delta j_\parallel \delta E_\parallel\rangle$ (Landau damping or collisions) contribute; however, those effects are omitted in the basic HM model.

Nevertheless, the $\gamma_{\boldsymbol{q}}$ arising from the interaction with the short scales does not vanish even for $q_\parallel=0$, as energy can be transferred from short to long perpendicular scales. To obtain that effect, one can develop a statistically averaged energy-conservation law or Poynting theorem that applies specifically to the short scales. To do so, multiply Eq. (100a) by $\varphi^>$, average over the short-scale statistics and integrate over the short spatial scales (jointly indicated by an overbar), and integrate by parts. Note that the short-scale averaging is *conditional* on the long-wavelength field or statistics. The integration by parts introduces surface terms, which we shall ignore in the following discussion. The first term arising from Eq. (100a) gives $\partial_t\overline{\mathcal{E}^>}$, where

$$\overline{\mathcal{E}^>}\doteq\frac{1}{2}[\overline{\alpha(\varphi^>)^2}+\overline{|\boldsymbol{\nabla}\varphi^>|^2}]. \tag{126}$$

If one had averaged as well over the long-wavelength statistics and ignored additive sources $\boldsymbol{\eta}$, then formula (126) would reduce to the standard result

$$\mathcal{E}=\langle\overline{\mathcal{E}^>}\rangle=\frac{1}{2}\sum_{\boldsymbol{k}}(\alpha_{\boldsymbol{k}}+k^2)C_{\boldsymbol{k}}=\sum_{\boldsymbol{k}}\sigma_{\boldsymbol{k}}^{(E)}C_{\boldsymbol{k}}; \tag{127}$$

however, in the absence of such averaging and in the presence of external sources, the statistical ensemble is inhomogeneous and it is preferable to work with the form (126). The second, linear diamagnetic term of Eq. (100a) does not contribute because of periodicity in the y direction. The third and fourth terms do not contribute because, after integration by parts, one obtains the construction $\boldsymbol{E}^>\cdot\boldsymbol{V}_E^>=0$. The fifth term, however, contributes

$$\overline{\varphi^>\boldsymbol{\nabla}\cdot[\boldsymbol{V}_E^<(-\nabla^2\varphi^>)]}=\overline{\boldsymbol{E}^>\cdot\boldsymbol{V}_E^<(-\nabla^2\varphi^>)} \tag{128a}$$

$$=-\boldsymbol{E}^<\cdot\overline{\boldsymbol{\Gamma}^>}, \tag{128b}$$

where the last result was obtained by interchanging a dot and cross product. We thus obtain the conditionally averaged Poynting theorem

$$\partial_t\overline{\mathcal{E}^>}=\boldsymbol{E}^<\cdot\overline{\boldsymbol{\Gamma}^>}+O(\hat{\eta}^>). \tag{129}$$

B. Long-wavelength growth from second-order energetics

Equation (129) can be used to obtain a new expression for the mean flux $\boldsymbol{\Gamma}\doteq\langle\overline{\boldsymbol{\Gamma}^>}\rangle$, whose first-order response is required in Eqs. (108b) or (109) for $\gamma_{\boldsymbol{q}}$. Again, we stress that Eq. (129) describes *conditionally averaged* statistics. To complete the statistical averaging, one must average Eq. (129) over the statistics of $\boldsymbol{E}^<$:

$$\partial_t\mathcal{E}=\langle\langle\boldsymbol{E}^<\rangle\rangle\cdot\boldsymbol{\Gamma}+\langle\langle\boldsymbol{E}^<\cdot\boldsymbol{\Gamma}\rangle\rangle+O(\hat{\eta}^>). \tag{130}$$

We shall ignore the cumulant $\langle\langle\boldsymbol{E}^<\cdot\boldsymbol{\Gamma}\rangle\rangle$, which contributes renormalization effects that can be inserted heuristically. (From the work in Sec. III, one already knows the correct, fully renormalized result.) Thus, we work with the statistically averaged Poynting theorem

$$\partial_t\mathcal{E}\approx\langle\langle\boldsymbol{E}^<\rangle\rangle\cdot\boldsymbol{\Gamma}+O(\hat{\eta}^>). \tag{131}$$

One can extract a formula for $\delta\boldsymbol{\Gamma}/\delta P^<\equiv\hat{\boldsymbol{\Gamma}}$ by *two* functional differentiations of Eq. (131). A first functional derivative with respect to $P^<(1')$ gives

$$\frac{\partial}{\partial t}\left(\frac{\delta\mathcal{E}(1)}{\delta P^<(1')}\right)\approx\langle\langle\boldsymbol{E}^<\rangle\rangle(1)\cdot\hat{\boldsymbol{\Gamma}}(1;1')-\boldsymbol{\nabla}\delta(1,1')\cdot\boldsymbol{\Gamma}(1). \tag{132}$$

Another derivative with respect to $P^<(1'')$ leads in the limit $\boldsymbol{\eta}\to 0$ to

$$\frac{\partial}{\partial t}\left(\frac{\delta^2\mathcal{E}(1)}{\delta P^<(1')\delta P^<(1'')}\right)\approx-[\boldsymbol{\nabla}\delta(1,1'')\cdot\hat{\boldsymbol{\Gamma}}(1;1')+(1'\Leftrightarrow 1'')]. \tag{133}$$

Upon defining $\hat{\hat{\mathcal{E}}}(1;1',1'')\doteq\delta^2\mathcal{E}(1)/\delta P^<(1')\delta P^<(1'')=\hat{\hat{\mathcal{E}}}(\underline{1}-\underline{1}',\underline{1}-\underline{1}'',t;t',t'')\to\hat{\hat{\mathcal{E}}}_{\boldsymbol{q}',\boldsymbol{q}''}(t;t',t'')$, one finds that the Fourier transform of Eq. (133) obeys

$$\partial_t\hat{\hat{\mathcal{E}}}_{\boldsymbol{q}',\boldsymbol{q}''}(t;t',t'')\approx-[i\boldsymbol{q}''\cdot\hat{\boldsymbol{\Gamma}}_{\boldsymbol{q}'}(t;t')\delta(t-t'')+i\boldsymbol{q}'\cdot\hat{\boldsymbol{\Gamma}}_{\boldsymbol{q}''}(t;t'')\delta(t-t')]. \tag{134}$$

Thus a first-order flux increment leads to a *second-order* energy variation. (The physics is no different from the familiar result that Ohmic heating is a second-order effect.) This suggests an alternative method of calculation in which one directly computes the second-order energy, then infers the first-order flux variation that defines $\gamma_{\boldsymbol{q}}$. This was the procedure used in the heuristic algorithm of Sec. II.

We now demonstrate in detail that the energy method leads to consistent results. Two procedures can be used to extract $i\boldsymbol{q}\cdot\hat{\boldsymbol{\Gamma}}_{\boldsymbol{q}}(t;t')$ from Eq. (134). One way is to set $\boldsymbol{q}'=\boldsymbol{q}''=\boldsymbol{q}$ and $t''=t'$; then, somewhat informally,

$$i\boldsymbol{q}\cdot\hat{\boldsymbol{\Gamma}}_{\boldsymbol{q}}(t;t')=-\frac{1}{2}\partial_t\hat{\hat{\mathcal{E}}}_{\boldsymbol{q},\boldsymbol{q}}(t;t',t')/\delta(t-t'). \quad (135)$$

However, this route is not the best: in addition to the formal difficulty of dividing by a Dirac δ function, the choice $\boldsymbol{q}'=\boldsymbol{q}''=\boldsymbol{q}$ corresponds to two differentiations with respect to $P_{\boldsymbol{q}}$ and leads to $\gamma_{\boldsymbol{q}}\propto+\partial_t\hat{\hat{\mathcal{E}}}_{\boldsymbol{q},\boldsymbol{q}}$ (with a counterintuitive sign), whereas from the calculations in Sec. II it appears more natural to differentiate once with respect to $P_{\boldsymbol{q}}$ and once with respect to $P^*_{\boldsymbol{q}}$. Therefore, a more intuitive and also more mathematically justifiable procedure is to order $t>t'>t''$ and to obtain the coefficient of the first δ function in Eq. (135) by integration. Upon setting $\boldsymbol{q}'=\boldsymbol{q}$ and $\boldsymbol{q}''=-\boldsymbol{q}$, one finds

$$i\boldsymbol{q}\cdot\hat{\boldsymbol{\Gamma}}_{\boldsymbol{q}}(t;t')=\int_{-\infty}^{t'-\epsilon}dt''\,\partial_t\hat{\hat{\mathcal{E}}}_{\boldsymbol{q},-\boldsymbol{q}}(t;t',t'') \quad (t>t') \quad (136)$$

Upon inserting this result into formula (108b), one finds

$$\gamma_{\boldsymbol{q}}\approx-\left(\frac{1}{\alpha_{\boldsymbol{q}}+q^2}\right)\int_{-\infty}^{t}dt'\int_{-\infty}^{t'-\epsilon}dt'' \times[\partial_t\hat{\hat{\mathcal{E}}}_{\boldsymbol{q},-\boldsymbol{q}}(t;t',t'')]R^*_{\boldsymbol{q}}(t;t'). \quad (137)$$

The overall minus sign in this formula is physically sensible, since $\hat{\hat{\mathcal{E}}}$ describes the second-order energy variation of the small scales whereas $\gamma_{\boldsymbol{q}}$ refers to the long wavelengths.

Let us show that formula (137) reduces to our previous result (112). An equation for $\hat{\hat{\mathcal{E}}}$ follows by differentiation of Eq. (114) with respect to $P^<(1'')$, Fourier transformation, and summation over the large $\boldsymbol{k}$'s. Upon differentiating Eq. (114) and setting to zero all mean-field and cross-correlation terms, one obtains

$$\begin{aligned}0=&(\alpha-\nabla^2)\partial_t\hat{\hat{C}}(1,2;1',1'')+V_*\partial_y\hat{\hat{C}}(1,2;1',1'')+\cdots\\&+\hat{\boldsymbol{\mathcal{V}}}(1,\overline{1})\hat{C}(\overline{1},2;1')\cdot\boldsymbol{\nabla}[-\nabla^2\delta(1,1'')]+(1'\Leftrightarrow1'')\\&+\hat{\boldsymbol{\mathcal{V}}}(1,1')\cdot\boldsymbol{\nabla}[-\nabla^2\hat{C}(1,2;1'')]+(1'\Leftrightarrow1''). \quad (138)\end{aligned}$$

With $\hat{\hat{C}}(\underline{1},t_1,\underline{2},t_2;\underline{1}',t',\underline{1}'',t'')=\hat{\hat{C}}(\underline{1}-\underline{2},t_1,t_2;\underline{1}-\underline{1}',t',\underline{1}-\underline{1}'',t'')\to\hat{\hat{C}}_{\boldsymbol{k};\boldsymbol{q}',\boldsymbol{q}''}(t_1,t_2;t',t'')$, Eq. (138) becomes

$$\begin{aligned}&\partial_{t_1}\hat{\hat{C}}_{\boldsymbol{k};\boldsymbol{q}',\boldsymbol{q}''}(t_1,t_2;t',t'')\\&=-i\Omega^{\mathrm{lin}}_{\boldsymbol{k}+\boldsymbol{q}'+\boldsymbol{q}''}\hat{\hat{C}}_{\boldsymbol{k};\boldsymbol{q}',\boldsymbol{q}''}+\frac{1}{\alpha+|\boldsymbol{k}+\boldsymbol{q}'+\boldsymbol{q}''|^2}\{\hat{\boldsymbol{z}}\cdot(\boldsymbol{k}+\boldsymbol{q}')\\&\quad\times\boldsymbol{q}''(q''^2-|\boldsymbol{k}+\boldsymbol{q}'|^2)\hat{C}_{\boldsymbol{k};\boldsymbol{q}'}(t_1,t_2;t')\delta(t-t'')\\&\quad+[(\boldsymbol{q}',t')\Leftrightarrow(\boldsymbol{q}'',t'')]\}. \quad (139)\end{aligned}$$

From $\partial_t\hat{\hat{C}}_{\boldsymbol{k};\boldsymbol{q}',\boldsymbol{q}''}(t,t;t',t'')=\partial_{t_1}\hat{\hat{C}}_{\boldsymbol{k};\boldsymbol{q}',\boldsymbol{q}''}(t_1,t_2;t',t'')|_{t_1=t_2=t}+[\boldsymbol{k}\to-(\boldsymbol{k}+\boldsymbol{q}'+\boldsymbol{q}'')]$, one obtains

$$\begin{aligned}\partial_t\hat{\hat{C}}_{\boldsymbol{k};\boldsymbol{q},-\boldsymbol{q}}(t,t;t',t'')=&(\alpha_{\boldsymbol{k}}+k^2)^{-1}\hat{\boldsymbol{z}}\cdot(\boldsymbol{q}\times\boldsymbol{k})\\&\times[(q^2-p^2)\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t;t')\\&-(q^2-|\boldsymbol{k}-\boldsymbol{q}|^2)\hat{C}_{-\boldsymbol{k};\boldsymbol{q}}(t;t')]\delta(t-t'')\\&+(\boldsymbol{q},t')\Leftrightarrow(-\boldsymbol{q},t''). \quad (140)\end{aligned}$$

(Note that the linear terms have canceled out.) Upon multiplying by $\frac{1}{2}(\alpha_{\boldsymbol{k}}+k^2)$ and summing over $\boldsymbol{k}$ to form the short-wavelength energy, one simplifies this to

$$\begin{aligned}\partial_t\hat{\hat{\mathcal{E}}}_{\boldsymbol{q},-\boldsymbol{q}}(t;t',t'')=&\sum_{\boldsymbol{k}}\hat{\boldsymbol{z}}\cdot(\boldsymbol{q}\times\boldsymbol{k})(q^2-p^2)\\&\times[\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t;t')\delta(t-t'')\\&+\hat{C}_{-\boldsymbol{k};-\boldsymbol{q}}(t;t'')\delta(t-t')]. \quad (141)\end{aligned}$$

According to Eq. (136), the coefficient of $\delta(t-t'')$ is $i\boldsymbol{q}\cdot\hat{\boldsymbol{\Gamma}}_{\boldsymbol{q}}(t;t')$. Thus formula (137) becomes

$$\gamma_{\boldsymbol{q}}=\left(\frac{1}{\alpha_{\boldsymbol{q}}+q^2}\right)\mathrm{Re}\sum_{\boldsymbol{k}}\hat{\boldsymbol{z}}\cdot(\boldsymbol{q}\times\boldsymbol{k})(p^2-q^2)\times\int_{-\infty}^{t}dt'\,\hat{C}_{\boldsymbol{k};\boldsymbol{q}}(t;t')R^*_{\boldsymbol{q}}(t;t'). \quad (142)$$

Now $p^2-q^2=k^2+2\boldsymbol{k}\cdot\boldsymbol{q}$. One can verify that the $\boldsymbol{k}\cdot\boldsymbol{q}$ term vanishes under the $\boldsymbol{k}$ summation by symmetry and reality; the resulting formula for $\gamma_{\boldsymbol{q}}$ is identical to Eq. (112), proving the equivalence of the energy method with the original definition based on first-order flux.

VI. CONDITIONAL AVERAGING AND THE WAVE KINETIC EQUATION

A feature of the calculations in both Secs. IV and V is that they all lead, though through somewhat different algebraic routes, to a general formula for $\gamma_{\boldsymbol{q}}$ that includes terms of all orders in ϵ. On the other hand, the heuristic algorithm presented in Sec. I C directly produces the lowest-order term in the ϵ expansion. It is of interest to understand why this happens and how various heuristic algorithms follow from the formal theory.

The fundamental distinction is that the formal methods of Secs. IV and V, involving unconditional averages over both short- and long-wavelength statistics, deal with a homogeneous statistical ensemble from the outset, whereas the heuristic algorithms (and a generalization thereof, a more formal conditional averaging procedure to be discussed below) refer to the response to a random potential and therefore work with a necessarily inhomogeneous ensemble (conditional on the statistics of $\tilde{\varphi}_{\boldsymbol{q}}$). The methods differ in the way in which inhomogeneity is exploited to obtain information about the statistics.

In the homogeneous ensemble, it is necessary to break the symmetry ''by hand,'' via the external sources $\boldsymbol{\eta}$, in order to enable functional derivation to probe successively finer and finer details of the statistics. Legendre transformation from $\boldsymbol{\eta}$ to $\langle\langle\boldsymbol{\varphi}\rangle\rangle$ enables one to close the statistical equations and to establish nontrivial functional relations between various quantities. However, because the functional variations refer to the exact nonlinear dynamics (which generate a spectrum of interactions involving all orders in ϵ), the final formulas also contain effects through all orders.

An alternative procedure is to average conditionally over the short scales, temporarily freezing the statistics of the long wavelengths. This procedure, which also breaks the symmetry, provides a different way of probing the system. One can examine the response of the short scales by a WKB analysis that proceeds order by order in ϵ. The lowest-order nontrivial description is the WKE correct through first order in the long-wavelength gradients. Note that there is no advantage to first averaging conditionally, developing a general description of inhomogeneous statistics, then finally averaging over the long-wavelength statistics; one will merely be led back to the functional relations in the homogeneous ensemble and to the general formula for $\gamma_{\boldsymbol{q}}$. The key to a reduced description is to truncate the dynamical content at the outset by working only to, say, first order in the WKB expansion.

A. WKB derivation of wave kinetic equations

WKB techniques for slightly inhomogeneous systems are well known. In the context of classical field theory, a fundamental reference is the work by Carnevale and Martin (CM) [37], who, in the course of a general discussion of Markovian statistical closures, attempted to derive WKE's for the energy and enstrophy evolution of Rossby waves in an inhomogeneous medium. Although their final equations obey reasonable conservation properties, close inspection reveals algebraic errors in their derivations. Furthermore, the general form of their spectral evolution equation [Eq. (150) below] does not agree with the one implied by the recent work of Smolyakov and Diamond [15], who used a (superficially) different method. In order to reconcile the various results, we shall therefore review and reconsider the development of CM. We will identify a subtle conceptual error in that otherwise excellent paper. When corrected, their results agree with those of Ref. [15].

As did CM, consider the treatment of the usual Dyson equations for second-order statistics in the presence of weak inhomogeneity. In this section, we will use a caret to distinguish abstract operators from their coordinate-dependent kernels. Then in operator form the Dyson equations are

$$\partial_t \hat{C} + \hat{\Sigma}\hat{C} = \hat{F}\hat{R}^\dagger, \tag{143a}$$

$$\partial_t \hat{R} + \hat{\Sigma}\hat{R} = \hat{1}. \tag{143b}$$

Here $\hat{\Sigma} \doteq i\hat{L} + \hat{\Sigma}^{\mathrm{nl}}$ includes both linear physics and the nonlinear coherent damping, and we have written F instead of F^{nl} to take account of possible external forcing. Operator products are realized as space-time convolutions: $(\hat{A}\hat{B})(1,1') = (A \star B)(1,1') \doteq \int d\overline{1}A(1,\overline{1})B(\overline{1},1')$. The procedure is to find an approximate representation for the convolution valid for small deviations from homogeneity. To illustrate with spatial coordinates only, we introduce

$$\boldsymbol{\rho} \doteq \boldsymbol{x} - \boldsymbol{x}', \quad \boldsymbol{X} \doteq \tfrac{1}{2}(\boldsymbol{x} + \boldsymbol{x}') \tag{144}$$

and write $A(\boldsymbol{x},\boldsymbol{x}') = A(\boldsymbol{\rho}|\boldsymbol{X})$. One now assumes that either A and/or B have a short correlation length in $\boldsymbol{\rho}$. Then expansion in small ρ/X led CM (incorrectly in general, as we will explain shortly) to

$$\begin{aligned}(\hat{A}\hat{B})(\boldsymbol{x},\boldsymbol{x}') \approx \int & d\overline{\boldsymbol{\rho}}A(\overline{\boldsymbol{\rho}}|\boldsymbol{X})B(\boldsymbol{\rho}-\overline{\boldsymbol{\rho}}|\boldsymbol{X}) \\ &+ \frac{1}{2}\left(\frac{\partial A(\overline{\boldsymbol{\rho}}|\boldsymbol{X})}{\partial \boldsymbol{X}} \cdot (\boldsymbol{\rho}-\overline{\boldsymbol{\rho}})B(\boldsymbol{\rho}-\overline{\boldsymbol{\rho}}|\boldsymbol{X}) \right. \\ &\left. - \overline{\boldsymbol{\rho}}A(\overline{\boldsymbol{\rho}}|\boldsymbol{X}) \cdot \frac{\partial B(\boldsymbol{\rho}-\overline{\boldsymbol{\rho}}|\boldsymbol{X})}{\partial \boldsymbol{X}}\right)\end{aligned} \tag{145}$$

or, upon Fourier transforming with respect to $\boldsymbol{\rho}$,

$$(\hat{A}\hat{B})_{\boldsymbol{k}}(\boldsymbol{X}) \approx A_{\boldsymbol{k}}(\boldsymbol{X})B_{\boldsymbol{k}}(\boldsymbol{X}) + \tfrac{1}{2}i\{A,B\}. \tag{146}$$

Here braces denote the Poisson bracket. Upon generalizing to include temporal variations, one defines the Poisson bracket of the two functions A and B (both with arguments $\boldsymbol{k}$, ω, $\boldsymbol{X}$, and T) as

$$\{A,B\} \doteq \left(\frac{\partial A}{\partial \boldsymbol{X}} \cdot \frac{\partial B}{\partial \boldsymbol{k}} - \frac{\partial A}{\partial \boldsymbol{k}} \cdot \frac{\partial B}{\partial \boldsymbol{X}}\right) - [(\boldsymbol{X},\boldsymbol{k}) \Leftrightarrow (T,\omega)]. \tag{147}$$

($\boldsymbol{X}$ and $\boldsymbol{k}$ play the roles of canonical coordinate and momentum, respectively, as do T and $-\omega$.) One is led to the Dyson equations in the form

$$\begin{aligned}\tfrac{1}{2}\partial_T C_{\boldsymbol{k},\omega}(\boldsymbol{X},T) &+ (-i\omega + \Sigma_{\boldsymbol{k},\omega})C_{\boldsymbol{k},\omega} \\ &= F_{\boldsymbol{k},\omega}R^*_{\boldsymbol{k},\omega} + \tfrac{1}{2}i(\{F_{\boldsymbol{k},\omega},R^*_{\boldsymbol{k},\omega}\} - \{\Sigma_{\boldsymbol{k},\omega},C_{\boldsymbol{k},\omega}\}),\end{aligned} \tag{148a}$$

$$(-i\omega + \Sigma_{\boldsymbol{k},\omega})R_{\boldsymbol{k},\omega}(\boldsymbol{X},T) + \tfrac{1}{2}i\{-i\omega + \Sigma_{\boldsymbol{k},\omega},R_{\boldsymbol{k},\omega}\} = 1. \tag{148b}$$

CM pointed out that under iterative solution of Eq. (148b) the Poisson-bracket term vanishes through first order, so

$$R^{-1}_{\boldsymbol{k},\omega}(\boldsymbol{X},T) \approx -i\omega + \Sigma_{\boldsymbol{k},\omega}(\boldsymbol{X},T), \tag{149}$$

correct through first order in the inhomogeneity.

The spectral balance equation for $C_{\boldsymbol{k}}(\boldsymbol{X},T)$ follows by taking the real part of Eq. (148a) and integrating over all ω's. The details are given in Ref. [37]. In the Markovian approximation for which the fluctuation-dissipation ansatz is invoked, one is led to a generalization of Eq. (15) that includes weak inhomogeneity. We will discuss the nonlinear terms in that equation elsewhere; for present purposes, it is sufficient to consider the explicitly linear terms. Upon approximating $\hat{\Sigma} \approx i\hat{L}$ and assuming for simplicity that $L = \Omega_{\boldsymbol{k}}(\boldsymbol{X},T)$ (a real function independent of ω), one obtains from Eq. (148a) [43]

$$\partial_T C_{\boldsymbol{k}}(\boldsymbol{X},T) - \{\Omega_{\boldsymbol{k}},C_{\boldsymbol{k}}\} \approx 0. \tag{150}$$

Explicitly,

$$-\{\Omega_{\boldsymbol{k}},C_{\boldsymbol{k}}\}=\frac{\partial\Omega_{\boldsymbol{k}}}{\partial\boldsymbol{k}}\cdot\frac{\partial C_{\boldsymbol{k}}}{\partial\boldsymbol{X}}-\frac{\partial\Omega_{\boldsymbol{k}}}{\partial\boldsymbol{X}}\cdot\frac{\partial C_{\boldsymbol{k}}}{\partial\boldsymbol{k}} \tag{151a}$$

$$=\frac{\partial}{\partial\boldsymbol{X}}\cdot\left(\frac{\partial\Omega_{\boldsymbol{k}}}{\partial\boldsymbol{k}}C_{\boldsymbol{k}}\right)-\frac{\partial}{\partial\boldsymbol{k}}\cdot\left(\frac{\partial\Omega_{\boldsymbol{k}}}{\partial\boldsymbol{X}}C_{\boldsymbol{k}}\right). \tag{151b}$$

This operator is the usual left-hand side of the WKE and is quoted ubiquitously in similar contexts.

Unfortunately, *Eq. (150) is incorrect in general*. The difficulty manifests itself when one attempts to construct the conservation laws associated with the statistical dynamics by summing Eq. (150) over $\boldsymbol{k}$ with appropriate weights. Thus, for pure HM or Rossby waves, equations for energy or enstrophy evolution follow by multiplying Eq. (150) by $\sigma_{\boldsymbol{k}}^{(E)}$ or $\sigma_{\boldsymbol{k}}^{(W)}$ and summing over $\boldsymbol{k}$. In so doing, one must not forget that because those weight factors depend on $\boldsymbol{k}$ they cannot be cavalierly moved in and out of the Poisson bracket. For example, with $\sigma_W\equiv\sigma_{\boldsymbol{k}}^{(W)}$,

$$\{\Omega_{\boldsymbol{k}},C_{\boldsymbol{k}}\}=\{\Omega_{\boldsymbol{k}},\sigma_W^{-1}W_{\boldsymbol{k}}\} \tag{152a}$$

$$=\sigma_W^{-1}\{\Omega_{\boldsymbol{k}},W_{\boldsymbol{k}}\}+\underline{\{\Omega_{\boldsymbol{k}},\sigma_W^{-1}\}W_{\boldsymbol{k}}}, \tag{152b}$$

or, as would appear in the equation for $\partial_t W_{\boldsymbol{k}}$,

$$\sigma_W\{\Omega_{\boldsymbol{k}},C_{\boldsymbol{k}}\}=\{\Omega_{\boldsymbol{k}},W_{\boldsymbol{k}}\}+\underline{\{\Omega_{\boldsymbol{k}},\ln\sigma_W^{-1}\}W_{\boldsymbol{k}}}. \tag{153}$$

CM mistakenly omitted the underlined correction term in their derivations of both the energy and enstrophy balances for Rossby waves. However, whereas the second, wave-number derivative term of Eq. (151b) vanishes upon integration over $\boldsymbol{k}$, the correction term does not. Thus, according to Eq. (150), the equation for, say, evolution of total enstrophy would be

$$\partial_T W(\boldsymbol{X},T)+\frac{\partial}{\partial\boldsymbol{X}}\cdot\left(\sum_{\boldsymbol{k}}\frac{\partial\Omega_{\boldsymbol{k}}}{\partial\boldsymbol{k}}W_{\boldsymbol{k}}\right)=\text{enstrophy-nonconserving term}. \tag{154}$$

For situations in which enstrophy is, in fact, conserved (as in the HM interaction between disparate scales studied below), Eq. (154) is evidently in error. That CM nevertheless obtained reasonable energy and enstrophy conservation laws in the face of an identifiable algebraic error is a symptom that their underlying WKE (150) is also incorrect.

The difficulty can be traced to a subtle error in the logic of the derivation of the weakly inhomogeneous convolution Eq. (146), rewritten here for the application $\hat{A}\to\hat{\Omega}$, $\hat{B}\to\hat{C}$:

$$\hat{\Omega}\hat{C}\approx\Omega C+\tfrac{1}{2}i\{\Omega,C\}. \tag{155}$$

Consider the case where $\hat{\Omega}$ is itself the product of two operators $\hat{A}$ and $\hat{B}$: $\hat{\Omega}=\hat{A}\hat{B}$. If $\hat{A}$ and $\hat{B}$ do not commute, $[\hat{A},\hat{B}]\doteq\hat{A}\hat{B}-\hat{B}\hat{A}\neq 0$, then the Fourier transform of the kernel of the product operator, $\Omega_{\boldsymbol{k},\omega}(\boldsymbol{X},T)$, contains terms of *first order* as well as of zeroth order in the gradients; the first-order contributions are incorrectly neglected in the derivation leading to Eq. (155). For an explicit example, consider for arbitrary function f the expression $f(\boldsymbol{x})\nabla_{\boldsymbol{x}}^2 C(\boldsymbol{x},\boldsymbol{x}')=(\hat{\Omega}\hat{C})(\boldsymbol{x},\boldsymbol{x}')$, for which

$$\Omega(\boldsymbol{x},\boldsymbol{x}')=-f(\boldsymbol{x})\nabla_{\boldsymbol{x}}^2\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{156}$$

One has

$$\Omega(\boldsymbol{\rho}|\boldsymbol{X})=-f(\boldsymbol{X}+\tfrac{1}{2}\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}^2\delta(\boldsymbol{\rho}) \tag{157a}$$

$$\approx-[f(\boldsymbol{X})+\tfrac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\nabla}f]\nabla_{\boldsymbol{\rho}}^2\delta(\boldsymbol{\rho}), \tag{157b}$$

or, upon Fourier transforming,

$$\Omega_{\boldsymbol{k}}(\boldsymbol{X})=f(\boldsymbol{X})k^2+\frac{1}{2}\boldsymbol{\nabla}f\cdot\frac{\partial(k^2)}{\partial(-i\boldsymbol{k})} \tag{158a}$$

$$=fk^2+\tfrac{1}{2}i\{f,k^2\} \tag{158b}$$

$$=fk^2+i\boldsymbol{k}\cdot\boldsymbol{\nabla}f. \tag{158c}$$

The contribution from the $\boldsymbol{k}\cdot\boldsymbol{\nabla}f$ term is absent from Eq. (155).

A systematic way of incorporating all first-order effects is to note that for the weakly inhomogeneous reduction of multiple convolutions the Poisson brackets between all pairs of operators must be included. This can be proved directly, but should be clear since otherwise the result will not respect the appropriate symmetry. For example,

$$\hat{A}\hat{B}\hat{C}\approx ABC+\tfrac{1}{2}i(\{A,B\}C+\{A,C\}B+\{B,C\}A), \tag{159}$$

where the quantities on the right-hand side are ordinary functions of $\boldsymbol{k}$ and $\boldsymbol{X}$ (and, in general, of ω and T). (Since such functions commute, the ones outside the Poisson brackets can be placed either to the left or the right.) With $\hat{\Omega}\doteq\hat{A}\hat{B}$, Eq. (159) can be rewritten as

$$\hat{\Omega}\hat{C}\approx\Omega C+\tfrac{1}{2}i(\{\Omega,C\}+\underline{\{A,B\}C}). \tag{160}$$

When applied to the above example (156) with $\hat{A}=f$, $\hat{B}=-\nabla^2$, the underlined correction term $\{A,B\}C$ reproduces the contribution from the last term of Eq. (158c).

B. Wave kinetic equations for pure and generalized Hasegawa-Mima dynamics

Let us now apply these considerations to the specific problem of the interactions of disparate scales in HM dynamics. To derive an approximate dynamical equation valid to lowest nontrivial order in the long-wavelength gradients, we neglect in Eq. (100a) the second term (it does not contribute to the WKE), the third term (short-wavelength renormalization effects), and the fourth term (of higher order in the long-wavelength gradients). For pure HM dynamics, we thus consider the dynamical equation

$$\partial_t\varphi+i\hat{\Omega}\varphi=0, \tag{161}$$

where

$$i\hat{\Omega} \doteq [(1-\nabla^2)^{-1}][\overline{\boldsymbol{V}}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}][-\nabla^2], \quad (162)$$

$\varphi \equiv \varphi^{>}$, $\overline{\boldsymbol{V}} \equiv \boldsymbol{V}_E^{<}$, and $\boldsymbol{\nabla}\cdot\overline{\boldsymbol{V}}=0$. Now $i\hat{\Omega}$ can be considered to be the product of the three operators delimited by the brackets in Eq. (162). (Strictly speaking, $\overline{\boldsymbol{V}}$ and $\boldsymbol{\nabla}$ are separate operators, but since they commute because $\overline{\boldsymbol{V}}$ has zero divergence, there is no need to consider them independently.) Thus $\hat{\Omega}=\hat{A}\hat{B}\hat{C}$, where

$$\hat{A} \doteq (1-\nabla^2)^{-1} \to (1+k^2)^{-1}, \quad (163a)$$

$$\hat{B} \doteq -i\overline{\boldsymbol{V}}\cdot\boldsymbol{\nabla} \to \boldsymbol{k}\cdot\overline{\boldsymbol{V}}(\boldsymbol{X},T), \quad (163b)$$

$$\hat{C} \doteq -\nabla^2 \to k^2. \quad (163c)$$

Upon temporarily writing the correlation function as D rather than C, one has

$$\hat{\Omega}\hat{D} \approx \Omega D + \tfrac{1}{2}i\{\Omega,D\} + \tfrac{1}{2}i(\{A,B\}C + \{A,C\}B + A\{B,C\})D, \quad (164)$$

where the zeroth-order frequency is

$$\Omega_{\boldsymbol{k}}(\boldsymbol{X},T) = \boldsymbol{k}\cdot\overline{\boldsymbol{V}}(\boldsymbol{X},T)k^2/(1+k^2). \quad (165)$$

Now $\{A,C\}=0$ [in general, $\{f(E),g(E)\}=0$, where f and g are arbitrary functions of an operator E; here both A and C are functions of k^2]; however, the Poisson brackets $\{A,B\}$ and $\{B,C\}$ contribute corrections because $\overline{\boldsymbol{V}}$ depends on $\boldsymbol{X}$. Thus

$$\{A,B\}C = -\frac{\partial}{\partial\boldsymbol{k}}\left(\frac{1}{1+k^2}\right)\cdot\boldsymbol{\nabla}(\boldsymbol{k}\cdot\overline{\boldsymbol{V}})k^2 \quad (166a)$$

$$= 2\boldsymbol{k}\cdot\boldsymbol{\nabla}\Omega_{\boldsymbol{k}}/(1+k^2) \quad (166b)$$

[Eq. (165) was used to rewrite the final result] and

$$A\{B,C\} = \left(\frac{1}{1+k^2}\right)\boldsymbol{\nabla}(\boldsymbol{k}\cdot\overline{\boldsymbol{V}})\cdot\frac{\partial(k^2)}{\partial\boldsymbol{k}} \quad (167a)$$

$$= 2\boldsymbol{k}\cdot\boldsymbol{\nabla}\Omega_{\boldsymbol{k}}/k^2. \quad (167b)$$

The coefficients of Eqs. (166b) and (167b) add as

$$\frac{1}{1+k^2} + \frac{1}{k^2} = \frac{1+2k^2}{k^2(1+k^2)} = \frac{1}{2k}\frac{\partial}{\partial k}\ln\sigma_W, \quad (168)$$

so the total correction to the basic wave kinetic Poisson bracket $\{\Omega,D\}$ can easily be found to be

$$\{\Omega,\ln\sigma_W\}D = -\{\Omega,\sigma_W^{-1}\}W. \quad (169)$$

This exactly cancels the enstrophy-nonconserving term obtained in Eqs. (152b)–(154), leading one to the final WKE

$$\partial_T C_{\boldsymbol{k}} - \sigma_W^{-1}\{\Omega_{\boldsymbol{k}},W_{\boldsymbol{k}}\} = 0 \quad (170a)$$

[compare Eq. (150)] or the trivially related one

$$\partial_T W_{\boldsymbol{k}} - \{\Omega_{\boldsymbol{k}},W_{\boldsymbol{k}}\} = 0. \quad (170b)$$

This last enstrophy-conserving equation agrees with a result of Smolyakov and Diamond [15] for pure HM dynamics. Those authors derived their formulas by working with a Fourier representation of the $\boldsymbol{X}$ dependence; the present $\boldsymbol{X}$-space method is arguably cleaner and easier to apply in general situations.

The same technique applies to the generalized HM problem. According to Eq. (14), the basic advection equation for the $k_\parallel \neq 0$ potential is

$$\partial_t\breve{\varphi} + (1-\nabla^2)^{-1}\overline{\boldsymbol{V}}\cdot\boldsymbol{\nabla}[(1-\nabla^2)\breve{\varphi}] = 0. \quad (171)$$

The basic frequency is therefore [15]

$$\Omega_{\boldsymbol{k}}(\boldsymbol{X},T) = \boldsymbol{k}\cdot\overline{\boldsymbol{V}}. \quad (172)$$

The operators $\hat{A}$ and $\hat{B}$ are as before, but now $\hat{C}=1-\nabla^2 = \hat{A}^{-1}$. One finds

$$\{A,B\}C = A\{B,C\} = \frac{2\boldsymbol{k}\cdot\boldsymbol{\nabla}\Omega_{\boldsymbol{k}}}{(1+k^2)^2}, \quad (173)$$

the total correction $\{\Omega,\ln\sigma_Z\}D$, and the Z-conserving WKE

$$\partial_T Z_{\boldsymbol{k}} - \{\Omega_{\boldsymbol{k}},Z_{\boldsymbol{k}}\} = 0. \quad (174)$$

Again, this agrees with a result of Ref. [15].

C. Physical interpretation of energy nonconservation: The significance of $\gamma_{\boldsymbol{k}}^{(1)}$

Although the appropriate invariant is conserved under the modulational interaction, energy is not; we illustrate for pure HM dynamics. Thus, upon multiplying Eq. (170a) by $\sigma^{(E)}$, one has

$$\partial_T E_{\boldsymbol{k}} - k^{-2}\{\Omega_{\boldsymbol{k}},k^2E_{\boldsymbol{k}}\} = 0 \quad (175)$$

or

$$\partial_T E_{\boldsymbol{k}} = \{\Omega_{\boldsymbol{k}},E_{\boldsymbol{k}}\} + 2\gamma_{\boldsymbol{k}}^{(1)}E_{\boldsymbol{k}}, \quad (176)$$

where

$$\gamma_{\boldsymbol{k}}^{(1)} \doteq \tfrac{1}{2}\{\Omega_{\boldsymbol{k}},\ln k^2\} = \boldsymbol{k}\cdot\boldsymbol{\nabla}\Omega_{\boldsymbol{k}}/k^2. \quad (177)$$

[The (1) superscript reminds one that $\gamma_{\boldsymbol{k}}^{(1)}$ is linearly proportional to $\overline{\boldsymbol{V}}$.] Although the first term on the right-hand side of Eq. (176) is now in conservation form, the $\gamma_{\boldsymbol{k}}^{(1)}$ term does not conserve energy.

A physical interpretation of $\gamma_{\boldsymbol{k}}^{(1)}$ follows by considering the ray equation for wave number $\boldsymbol{k}$ in a weakly inhomogeneous medium governed by frequency $\Omega_{\boldsymbol{k}}(\boldsymbol{X})$:

$$d\boldsymbol{k}/dt = -\boldsymbol{\nabla}\Omega_{\boldsymbol{k}}. \quad (178)$$

Then

$$d(k^2)/dt = -2\boldsymbol{k}\cdot\boldsymbol{\nabla}\Omega_{\boldsymbol{k}} = -2\gamma_{\boldsymbol{k}}^{(1)}k^2. \quad (179)$$

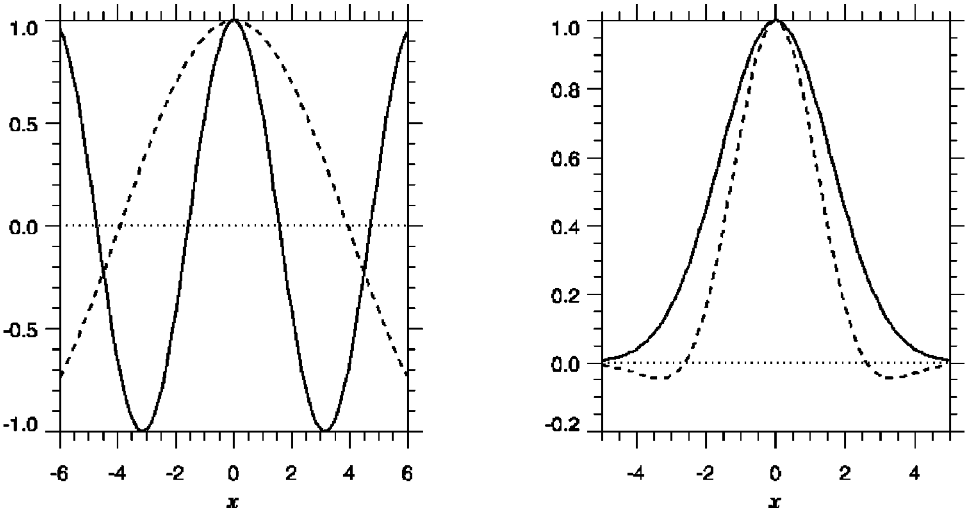

FIG. 7. Illustration of the mechanism underlying $\gamma_{\bm{k}}^{(1)}$ and the ray equation (178). On the left is plotted $\cos[kx-\Omega(x)t]$ for $\Omega(x)=x$ at $t=0$ (solid line) and $t=0.6$ (dashed line); the effective wave number decreases as t increases. On the right is the result of a Gaussian wave-number superposition of such cosines. The spatial narrowing of the wave packet is not properly described as shearing.

$\gamma_{\bm{k}}^{(1)}$ thus describes the logarithmic rate of wave-number evolution due to the inhomogeneity induced by the long-wavelength modulation. This same result follows by considering the potential

$$\varphi(\bm{x},t)=\varphi_0\cos[\bm{k}\cdot\bm{x}-\Omega(\bm{x})t] \tag{180}$$

and calculating the mean-square gradient averaged over the initial wavelength. Since $\nabla\varphi=-(\bm{k}-t\nabla\Omega)\varphi_0\sin(\bm{k}\cdot\bm{x}-\Omega_{\bm{k}}t)$, one finds that through first order in the gradient

$$\overline{k^2}\doteq\overline{|\nabla\varphi|^2}/\overline{\varphi^2}=-2\bm{k}\cdot\nabla\Omega t, \tag{181}$$

which reproduces Eq. (179). The effect is illustrated in Fig. 7. Note that it exists for $\bm{k}$-independent Ω, i.e., for vanishing group velocity.

Now since enstrophy W is conserved and $W_{\bm{k}}=k^2E_{\bm{k}}$, one has heuristically

$$\dot{W}_{\bm{k}}=0=(\dot{k^2})E_{\bm{k}}+(k^2)\dot{E}_{\bm{k}}, \tag{182}$$

or

$$\dot{E}_{\bm{k}}=2\gamma_{\bm{k}}^{(1)}E_{\bm{k}}. \tag{183}$$

More formally, this last result is to be understood in the sense of the integration of Eq. (176) over $\bm{k}$ and $\bm{X}$:

$$\partial_T\overline{E}=2\sum_{\bm{k}}\overline{\gamma_{\bm{k}}^{(1)}E_{\bm{k}}}. \tag{184}$$

We must emphasize that the physics content of this discussion is entirely compatible with the one given by Kraichnan [5] in his interpretation of negative eddy viscosity in 2D. In particular, Kraichnan recognized the importance of wave-number evolution, although he studied a particular example. Our modest contribution is to give a more general discussion that shows how that effect arises in the general context of WKE's. Also note that if the calculations of this section are repeated for generalized HM dynamics the only changes are to replace k^2 by $\overline{k^2}$ and W by Z.

D. Justification of the heuristic energy algorithm for $\gamma_{\bm{q}}$

These considerations provide the justification for the heuristic energy algorithm described in Sec. II. Formula (29) follows from Eq. (137). A virtue of the formal derivation is that the nonlinear mode-mode interaction time is handled systematically; cf. the presence of $R_{\bm{q}}$ in Eq. (137). For heuristic manipulations, one may introduce that by hand. For example, for the contribution of one long-wavelength Fourier component $\bm{q}$ to the enstrophy evolution, Eq. (170b) generalizes to

$$\partial_TW_{\bm{k}}+(\theta^{r}_{\bm{k},-\bm{k},\bm{q}})^{-1}W_{\bm{k}}=\{\tilde{\Omega}_{\bm{k}},W_{\bm{k}}\}, \tag{185}$$

where we now recognize that the $\overline{\bm{V}}\equiv\bm{V}_E^{<}$ in Eq. (165) is created by the random long-wavelength potential, so we write $\tilde{\Omega}_{\bm{k}}$ instead of $\Omega_{\bm{k}}$. (We specifically do not include $\Omega_{\bm{k}}^{\mathrm{lin}}$ in view of the discussion in Sec. III C of the $qv_{\mathrm{gr},\bm{k}}^{\mathrm{lin}}$ term.) The evolution of the short-wavelength energy $\mathcal{E}$ (summed over large $\bm{k}$ and integrated over $\bm{X}$) can be obtained most conveniently from Eq. (176):

$$\partial_T\mathcal{E}=2\sum_{\bm{k}}\overline{\tilde{\gamma}_{\bm{k}}^{(1)}E_{\bm{k}}}, \tag{186}$$

although exactly the same result could be obtained from Eq. (170b) by dividing by k^2, then integrating by parts under the $\bm{k}$ sum (as was done in Sec. II). If one notes that $\tilde{\Omega}$ is of first order in $\tilde{\varphi}^{<}$, it is clear that the second-order variation of Eq. (186) obeys

$$\partial_T\hat{\hat{\mathcal{E}}}=2\sum_{\bm{k}}(\bm{k}\cdot\nabla\hat{\Omega}_{\bm{k};\bm{q}}/k^2)^*\hat{E}_{\bm{k};\bm{q}} \tag{187a}$$

$$=2\sum_{\bm{k}}\frac{1}{k^4}(\bm{k}\cdot\nabla\hat{\Omega}_{\bm{k};\bm{q}})^*\hat{W}_{\bm{k};\bm{q}}. \tag{187b}$$

Since Eqs. (176) and (170b) are equivalent, one may use either to calculate the first-order variations $\hat{E}$ or $\hat{W}$; it is most convenient to use Eq. (170b). The associated algebra was performed in Sec. II; the only difference is that, whereas there we heuristically asserted that one should integrate the time-dependent enstrophy equation (38) over the mode-mode interaction time $\theta^{r}_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$, more systematically one should solve Eq. (185) in steady state.

E. Fokker-Planck interpretation of the short-scale spectral evolution

Both the conservation of $Q_{\boldsymbol{k}}$ under long-wavelength advection as well as the form of Eq. (66) suggest that a Fokker-Planck description of the large-$\boldsymbol{k}$ wave packets is appropriate, with $Q_{\boldsymbol{k}}$ playing the role of a probability density function for $\boldsymbol{k}$. The general form of a Fokker-Planck equation is

$$\frac{\partial Q_{\boldsymbol{k}}}{\partial T} = -\frac{\partial}{\partial \boldsymbol{k}}\cdot(\boldsymbol{V}_{\boldsymbol{k}}Q_{\boldsymbol{k}}) + \frac{\partial}{\partial \boldsymbol{k}}\frac{\partial}{\partial \boldsymbol{k}}:(\mathsf{D}_{\boldsymbol{k}}Q_{\boldsymbol{k}}), \tag{188}$$

where

$$\boldsymbol{V}_{\boldsymbol{k}} \doteq \lim_{\Delta T\to ``0"}\langle\Delta\boldsymbol{k}\rangle/\Delta T$$

and $\mathsf{D}_{\boldsymbol{k}} \doteq \lim_{\Delta T\to ``0"}\langle\Delta\boldsymbol{k}\Delta\boldsymbol{k}\rangle/2\Delta T$, $\Delta\boldsymbol{k}$ being the increment of the random Lagrangian wave number in a short time ΔT. The quotes in the notation $\Delta T\to$ ''0'' mean that ΔT must remain larger than the appropriate autocorrelation time, which in this case is $\theta^{r}_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$. Now in the presence of the slowly varying advection frequency $\widetilde{\Omega}_{\boldsymbol{k}}$ one has the ray equations (Hamiltonian in form, with $\widetilde{\Omega}_{\boldsymbol{k}}$ playing the role of H)

$$\frac{d\boldsymbol{X}}{dT} = \frac{\partial\widetilde{\Omega}_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \equiv \widetilde{\boldsymbol{V}}_{\mathrm{gr},\boldsymbol{k}}, \tag{189a}$$

$$\frac{d\boldsymbol{k}}{dT} = -\boldsymbol{\nabla}\widetilde{\Omega}_{\boldsymbol{k}}. \tag{189b}$$

Because $\widetilde{\Omega}_{\boldsymbol{k}}$ is random and has zero mean, $\boldsymbol{X}$ does not vary to zeroth order, so correct to first order one has

$$\Delta\boldsymbol{k}^{(1)} = -\int_{T}^{T+\Delta T} dT'\,\boldsymbol{\nabla}\widetilde{\Omega}_{\boldsymbol{k}}(\boldsymbol{X},T'). \tag{190}$$

Upon averaging $\Delta\boldsymbol{k}^{(1)}\Delta\boldsymbol{k}^{(1)}$ over $\boldsymbol{X}$ as well as the statistics of $\widetilde{\varphi}_{\boldsymbol{q}}$, one readily finds that $\mathsf{D}_{\boldsymbol{k}}$ is given by Eq. (67a). Upon working out the second-order contribution to $\langle\Delta\boldsymbol{k}\rangle$ from the first-order correction to $\boldsymbol{X}$ due to wave-packet propagation described by Eq. (189a), one finds a contribution $\Delta\boldsymbol{V}_{\boldsymbol{k}} = \partial_{\boldsymbol{k}}\cdot\mathsf{D}_{\boldsymbol{k}}$, which when subtracted from the last term of Eq. (188) converts it to standard diffusion form [the first term of Eq. (66)]; this is a well-known consequence of Hamiltonian dynamics. Finally, we expand the total time derivative in Eq. (189b) to nonlinear order,

$$\frac{d}{dT} = \frac{\partial}{\partial T} + \widetilde{\boldsymbol{V}}_{\mathrm{gr},\boldsymbol{k}}\cdot\boldsymbol{\nabla}, \tag{191}$$

and find the second-order contribution to $\langle\Delta\boldsymbol{k}\rangle$ due to wave-packet advection to be

$$\langle\Delta\boldsymbol{k}^{(2)}\rangle = -\int_{T}^{T+\Delta T} dT'\langle\widetilde{\boldsymbol{V}}^{(1)}_{\mathrm{gr},\boldsymbol{k}}(T')\cdot\boldsymbol{\nabla}\boldsymbol{k}^{(1)}(T')\rangle. \tag{192}$$

Upon calculating the resulting $V_{\boldsymbol{k}}$, one is led to the second, drag term in Eq. (66). [Note that the potentials that contribute in mean square to Eq. (192) are those of the *small* scales.] It can be shown that the diffusion and drag terms also follow from detailed statistical analysis of the random WKE for $Q_{\boldsymbol{k}}$. It was already noted in Ref. [2] that a ''quasilinear'' analysis of that equation led to the wave-number diffusion effect. The present calculations extend that analysis to include the proper mode-mode interaction time and the drag effect; they highlight the roles of the random ray equation (189b) and the first-order distension rate $\widetilde{\gamma}^{(1)}_{\boldsymbol{k}}$.

A useful analogy is to the classical plasma collision operators $C_{ss'}$ for electron-ion and ion-electron scattering. C_{ei} and C_{ie} are not individually energy conserving, but do conserve kinetic energy when summed over species. Our expansion parameter $\epsilon \doteq q/k$ is analogous to the small parameter m_e/m_i in the classical problem, with small q's being analogous to light electrons and large k's being analogous to heavy ions. In classical kinetic theory, C_{ie} obeys a Fokker-Planck equation just as does $Q_{\boldsymbol{k}}$ in the present problem. The $\mathsf{D}_{\boldsymbol{k}}$ term in Eq. (66) is analogous to velocity-space diffusion; the $\boldsymbol{\Gamma}$ term is analogous to polarization drag. Interactions of comparable scale, which we do not study in this paper, are analogous to C_{ee} and C_{ii}. Some more detailed discussion of such analogies, including the relationship of incoherent noise to polarization drag, was given in Ref. [44].

F. A heuristic algorithm based on first-order flux

Finally, we shall describe a heuristic algorithm based on first-order variation of the flux. Because there are some subtle points, it is useful to first discuss some issues relating to *particle* transport in the *dissipative* HM system, without worrying about the functional variation that must be taken to obtain the ultimate answer for the nonlinear $\gamma_{\boldsymbol{q}}$. The generalization to the nonlinear vorticity transport that determines $\gamma_{\boldsymbol{q}}$ will then be straightforward.

Consider the HM system in the presence of nonadiabatic electron response. For convenience, we repeat that here:

$$\partial_t n_i^{\mathrm{G}} + V_* \partial_y \varphi + \boldsymbol{V}_E\cdot\boldsymbol{\nabla} n_i^{\mathrm{G}} = 0, \tag{193a}$$

$$n_i^{\mathrm{G}} + n_i^{\mathrm{pol}} = n_e = (1 - i\hat{\delta})\varphi, \tag{193b}$$

where $n_i^{\mathrm{pol}} \doteq \nabla^2\varphi$ and all variables represent fluctuations. [We are not concerned here with the nonadiabatic response of $k_\parallel = 0$ modes, so we merely write a 1 rather than $\hat{\alpha}$ on the right-hand side of Eq. (193b).] In Eq. (193b), $\hat{\delta}$ is a time-independent operator in real space whose Fourier transform is $\delta_{\boldsymbol{k}} > 0$. For small $\delta_{\boldsymbol{k}}$, the system (193) supports a normal mode at frequency $\omega^{\mathrm{lin}}_{\boldsymbol{k}} = \Omega^{\mathrm{lin}}_{\boldsymbol{k}} + i\gamma^{\mathrm{lin}}_{\boldsymbol{k}}$, where $\Omega^{\mathrm{lin}}_{\boldsymbol{k}}$ is approximately given by Eq. (7a) and

$$\gamma^{\mathrm{lin}}_{\boldsymbol{k}}/\Omega^{\mathrm{lin}}_{\boldsymbol{k}} \approx \delta_{\boldsymbol{k}}/(1+k^2). \tag{194}$$

With the aid of Poisson's equation (193b), one can easily show that the mean gyrocenter fluxes for either electrons or ions are equal; that is, the gyrocenter transport is intrinsically ambipolar on the average. We are interested in several different ways of obtaining the formula for the unique flux Γ. In method 1, we make a direct calculation of Γ_e (and also use the GK Poisson constraint to show that $\Gamma_i = \Gamma_e$). In method 2, we calculate Γ_i by first introducing the ion gyrocenter susceptibility, a technique employed in Refs. [45] and [2]. There are instructive subtleties in this latter approach.

In general, one has

$$\Gamma_s^G = \langle V_{E,x}(\boldsymbol{x},t) n_s^G(\boldsymbol{x},t)\rangle. \tag{195}$$

Almost invariably this function is taken to be independent of fast space and time, so that Fourier representations can be introduced. Thus

$$\Gamma_s^G = i\sum_{\boldsymbol{k}} k_y \langle n_{s,\boldsymbol{k}}^G(t)\varphi_{\boldsymbol{k}}^*(t)\rangle \tag{196a}$$

$$= i\sum_{\boldsymbol{k}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} k_y \langle n_{s,\boldsymbol{k}}^G \varphi_{\boldsymbol{k}}^*\rangle(\omega). \tag{196b}$$

In Eq. (196b), ω is real. The notation $\langle A\, B\rangle(\omega)$ means the Fourier transform with respect to τ of $\langle A(t+\tau)B(t)\rangle$.

For the electron flux in method 1, one simply substitutes $\delta n_{e,\boldsymbol{k}} = (1 - i\,\delta_{\boldsymbol{k}})\varphi_{\boldsymbol{k}}$ into Eq. (196a), obtaining

$$\Gamma_e^G = \sum_{\boldsymbol{k}} k_y \delta_{\boldsymbol{k}} C_{\boldsymbol{k}}(T), \tag{197}$$

where in steady state $C_{\boldsymbol{k}}(T)$ would be independent of the slow time T. The same result is obtained for Γ_i^G if one substitutes $\delta n_i^G = -\delta n_i^{\rm pol} + \delta n_e$, because the ion polarization density $-k^2\delta\varphi_{\boldsymbol{k},\omega}$ is in phase with the potential and does not contribute on the average. Observe that nowhere in this calculation did the ion gyrocenter dynamics or the properties of the linear normal mode of the HM system enter. Of course, the fluctuation spectrum is not yet determined. But given that spectrum, the electron flux obtains its value merely by virtue of the specified phase shift between electron density and potential, and the ion flux realizes that same value by virtue of the constraint enforced by the GK Poisson equation. These well-known results are true for arbitrarily large nonlinearity.

Now consider method 2. Suppose one introduces the ion gyrocenter susceptibility $\chi_i^G(\boldsymbol{k},\omega)$ such that

$$n_{i,\boldsymbol{k}}^G(\omega) = -\chi_i^G(\boldsymbol{k},\omega)\varphi_{\boldsymbol{k}}(\omega). \tag{198}$$

Because $\chi_i^G(\boldsymbol{k},\omega)$ is a causal response function, it is analytic in the upper half of the ω plane. The fully nonlinear susceptibility is very difficult to compute [19]. However, in linear theory one readily finds

$$\chi_i^G(\boldsymbol{k},\omega) = -\left(\frac{\omega_*}{\omega + i\epsilon}\right) = -\mathrm{P}\left(\frac{\omega_*}{\omega}\right) + i\pi\omega_*\delta(\omega), \tag{199}$$

where P denotes the principal value and the positive infinitesimal ϵ ensures causality. (It is a common misconception that the drift-wave linear ion susceptibility is real.) In order to use this frequency-dependent function for flux calculations, one must use the frequency-resolved form (196b). Thus

$$\Gamma_i^G = \sum_{\boldsymbol{k}} \int \frac{d\omega}{2\pi} k_y \,\mathrm{Im}\,\chi_i(\boldsymbol{k},\omega) C_{\boldsymbol{k}}(\omega). \tag{200}$$

It is important to understand that formula (200) is not a decomposition into normal modes. Indeed, in linear theory the imaginary part of the susceptibility contributes only at zero frequency: $\mathrm{Im}\,\chi_i^G(\boldsymbol{k},\omega) = \pi\omega_*\delta(\omega)$. Thus

$$\Gamma_i^G = \frac{1}{2}\sum_{\boldsymbol{k}} k_y \omega_*(\boldsymbol{k}) C_{\boldsymbol{k}}(\omega = 0). \tag{201}$$

A useful and physically significant alternative form of Eq. (201) follows by noting that for stationary correlation functions $C(\tau)$, which depend only on time difference, an autocorrelation time is conventionally defined by $\tau_{\rm ac} = C^{-1}\int_0^\infty d\tau\, C(\tau) = \frac{1}{2}C(\omega=0)/C$, where $C \equiv C(\tau=0)$. Thus the Eulerian, $\boldsymbol{k}$-dependent autocorrelation time is

$$\tau_{{\rm ac},\boldsymbol{k}} \doteq C_{\boldsymbol{k}}(\omega=0)/2C_{\boldsymbol{k}}, \tag{202}$$

and formula (201) becomes

$$\Gamma_i^G = \sum_{\boldsymbol{k}} k_y [\omega_*(\boldsymbol{k})\tau_{{\rm ac},\boldsymbol{k}}] C_{\boldsymbol{k}}. \tag{203}$$

(One can verify that this corresponds to a turbulent diffusion coefficient with the standard random-walk scaling $D \sim \langle \delta \boldsymbol{V}_E^2\rangle \tau_{\rm ac}$.)

Formulas (201) and (203) are not in obvious agreement with Eq. (197), but we have not yet determined the autocorrelation time or equivalently the spectral intensity at $\omega = 0$. To do so, we examine the constraint imposed by Poisson's equation (193b):

$$-\chi_i^G(\boldsymbol{k},\omega) C_{\boldsymbol{k}}(\omega) = (1 + k^2 - i\,\delta_{\boldsymbol{k}}) C_{\boldsymbol{k}}(\omega). \tag{204}$$

Upon integrating the imaginary part of this equation over all ω's and noting that $(2\pi)^{-1}\int_{-\infty}^{\infty} d\omega\, C_{\boldsymbol{k}}(\omega) = C_{\boldsymbol{k}}$, one obtains

$$-\frac{1}{2}\omega_* C_{\boldsymbol{k}}(\omega=0) = -\delta_{\boldsymbol{k}} C_{\boldsymbol{k}}. \tag{205}$$

Thus the rigid constraint imposed by Poisson's equation fully determines the autocorrelation time:

$$\tau_{{\rm ac},\boldsymbol{k}} = \delta_{\boldsymbol{k}}/\omega_*. \tag{206}$$

This is exactly what is required in order to bring Eqs. (203) and (197) into agreement. It also has an important physical interpretation. Recall that for a correlation function with frequency Ω and weak damping $\eta \ll \Omega$ one has

$$\tau_{\rm ac} = \mathrm{Re}\int_0^\infty d\tau\, e^{-i\Omega\tau - \eta\tau} = \frac{\eta}{\Omega^2 + \eta^2} \approx \frac{\eta}{\Omega^2} \quad (\eta/\Omega \ll 1). \tag{207}$$

With $\eta \to \gamma_{\boldsymbol{k}}^{\rm lin}$ and $\Omega \to \Omega_{\boldsymbol{k}}^{\rm lin}$, formula (207) reduces with the aid of Eq. (194) to the result (206).

Returning now to the interactions of disparate scales, one can use formula (203) to give an alternative heuristic derivation of the long-wavelength growth rate by replacing

$\omega_*/\overline{k^2}=\Omega^{\rm lin}$ by the advection frequency $\tilde{\Omega}$ and using an appropriate autocorrelation time. It is natural to build $\tau_{\mathrm{ac},\boldsymbol{k}}$ from $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}\doteq\gamma_{\boldsymbol{k}}^{(1)}[\tilde{\Omega}]$ and $\tilde{\Omega}$. For definiteness, we consider pure HM dynamics. Then $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}/\tilde{\Omega}=\boldsymbol{k}\cdot\boldsymbol{\nabla}\ln\tilde{\Omega}/k^2\ll 1$, which is the proper limit for the use of Eq. (207). Replace η by $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$, Ω by $\tilde{\Omega}$, and the gradients by $i\boldsymbol{q}$. Then note that

$$\tilde{\Omega}\tilde{\tau}_{\mathrm{ac},\boldsymbol{k}}=\tilde{\gamma}_{\boldsymbol{k}}^{(1)}/\tilde{\Omega}\to i\boldsymbol{k}\cdot\boldsymbol{q}/k^2=iqk_x/k^2 \tag{208}$$

is independent of $\tilde{\varphi}_{\boldsymbol{q}}$, so its functional variation vanishes. Now Eq. (109) states that

$$\gamma_{\boldsymbol{q}}=-(\alpha_{\boldsymbol{q}}+q^2)^{-1}i\boldsymbol{q}\cdot\hat{\boldsymbol{\Gamma}}_{\boldsymbol{q}}, \tag{209}$$

where $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{q}}$ describes the transport of *vorticity* $k^2\varphi$, not ion gyrocenter density. From Poisson's equation, one has $k^2\varphi=(k^2/\overline{k^2})n_i^G$, so, for use in Eq. (209), Eq. (203) must be corrected by the ratio $k^2/\overline{k^2}$:

$$\hat{\boldsymbol{\Gamma}}_{\boldsymbol{q}}=\hat{\boldsymbol{q}}\sum_{\boldsymbol{k}} k_y(\overline{k^2}\tilde{\Omega}\tilde{\tau}_{\mathrm{ac},\boldsymbol{k}})\left(\frac{k^2}{\overline{k^2}}\right)\hat{C}_{\boldsymbol{k},\boldsymbol{q}}. \tag{210}$$

Upon combining Eqs. (208)–(210), one obtains

$$\gamma_{\boldsymbol{q}}=\left(\frac{q^2}{\alpha_{\boldsymbol{q}}+q^2}\right)\sum_{\boldsymbol{k}} k_yk_x\left(\frac{\overline{k^2}}{k^2}\right)\hat{C}_{\boldsymbol{k};\boldsymbol{q}}. \tag{211}$$

The first-order variation $\hat{C}_{\boldsymbol{k};\boldsymbol{q}}$ may be calculated from any of Eqs. (170a), (170b), or (176) in the standard way [cf. Eq. (39)]; one again recovers Eq. (22b).

This algorithm depends only on the nonlinear ($\tilde{\varphi}$-dependent) quantities $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$ and $\tilde{\Omega}$, as has been the case for all of the algorithms discussed in this article. The present discussion suggests how one might be misled into introducing the properties of *linear* modes into a heuristic algorithm, since $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}/\tilde{\Omega}$ is independent of $\tilde{\varphi}_{\boldsymbol{q}}$. However, our systematic derivations from renormalized field theory show that those properties are irrelevant. Thus it is physically unjustifiable to obtain the factor of k_x in Eq. (211) from the relation

$$k_x\propto\left(\frac{\partial\mathcal{D}(\boldsymbol{k},\omega)}{\partial\boldsymbol{k}}\right)\Bigg|_{\omega}=-\left(\frac{\partial\omega}{\partial\boldsymbol{k}}\right)\Bigg|_{\mathcal{D}}\left(\frac{\partial\mathcal{D}(\boldsymbol{k},\omega)}{\partial\omega}\right)\Bigg|_{\boldsymbol{k}} \tag{212}$$

(where $\mathcal{D}$ is the linear dielectric function), as was suggested in Ref. [2]. The physics content of this formula is an asserted balance between wave-packet propagation and linear normal-mode energy growth. However, in fact the k_x arises from $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$, the quite different physics content of which is the ray equation (189b). The origin of Eq. (212) is an incomplete energy-balance equation used in Ref. [45], as discussed in Sec. VII.

We conclude this section with some further remarks about the frequency-resolved flux formula (196b). We emphasize again that the calculations leading to Eq. (206) did not invoke normal modes (whose properties are determined by the real part of the dielectric response). But suppose one insisted that, in fact, the shape of the spectrum *is* determined by the normal modes. For a spectrum with a single eigenmode at $\omega\approx\Omega_{\boldsymbol{k}}$, a typical Lorentzian form would be

$$C_{\boldsymbol{k}}(\omega)\approx\frac{1}{\pi}\left(\frac{\nu_{\boldsymbol{k}}}{(\omega-\Omega_{\boldsymbol{k}})^2+\nu_{\boldsymbol{k}}^2}\right)2\pi C_{\boldsymbol{k}}, \tag{213}$$

where $\nu_{\boldsymbol{k}}$ must be determined. If the eigenmode is to be well formed, one must have $\nu_{\boldsymbol{k}}/\Omega_{\boldsymbol{k}}\ll 1$. Then

$$\tau_{\mathrm{ac},\boldsymbol{k}}=\frac{C_{\boldsymbol{k}}(\omega=0)}{2C_{\boldsymbol{k}}}\approx\frac{\nu_{\boldsymbol{k}}}{\Omega_{\boldsymbol{k}}^2}. \tag{214}$$

According to Eq. (194), this will agree with the exact result (206) if one chooses $\nu_{\boldsymbol{k}}=\gamma_{\boldsymbol{k}}^{\rm lin}$ (correct only in the limit of small $\gamma_{\boldsymbol{k}}^{\rm lin}/\Omega_{\boldsymbol{k}}$).

This result might appear to be intuitively obvious. However, so far the susceptibility calculations have invoked only properties of linear theory, but *linear theory cannot be in steady state*. Forms like Eq. (213) are really appropriate only in statistical steady state, in which (i) the precise form of $C_{\boldsymbol{k}}(\omega)$ is not known, and (ii) $\nu_{\boldsymbol{k}}$ is a measure of nonlinear decorrelation processes—not, intrinsically, linear growth. Now the system (193) will not achieve steady state unless ion damping is added to the dynamics. Assume that has been done. The exact value of the steady-state flux is still Eq. (197). Let us enquire whether properties of the statistical steady state can be used to reconcile the values of Γ_i^G and Γ_e^G. The general statistical balance equation can be written as

$$C_{\boldsymbol{k}}(\omega)=\frac{\tilde{C}_{\boldsymbol{k}}(\omega)}{|\mathcal{D}(\boldsymbol{k},\omega)|^2}, \tag{215}$$

where $\tilde{C}$ describes nonlinear incoherent noise. A constraint on the spectral intensity is obtained by multiplying Eq. (215) by $\mathcal{D}(\boldsymbol{k},\omega)$:

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}[\chi_i^G(\boldsymbol{k},\omega)+\chi_e^G(\boldsymbol{k},\omega)]C_{\boldsymbol{k}}(\omega)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\tilde{C}_{\boldsymbol{k}}(\omega)}{\mathcal{D}^*(\boldsymbol{k},\omega)}. \tag{216}$$

If the right-hand side were to vanish, the resulting constraint would reconcile the steady-state ion and electron fluxes. But it does not vanish; $\tilde{C}_{\boldsymbol{k}}(\omega)$ both is a positive-definite form and, being the Fourier transform of a two-sided covariance, has structure in both halves of the complex ω plane. Therefore the nonlinear steady-state theory might appear to violate the known constraint of ambipolarity. However, this paradox can be easily resolved: The susceptibilities, even in their fully nonlinear forms, are response functions that describe *infinitesimal* perturbations [19] away from steady state, whereas the density fluctuations that appear in the flux formulas are the actual, finite-sized ones *in* the steady state. Density is not related to potential via a first-order infinitesimal response function. Therefore, it is inappropriate to use Eq. (198) anywhere except for linear response. And then the Poisson-equation constraint on the spectrum guarantees ambipolarity.

In the application of these ideas to the calculation of $\gamma_{\boldsymbol{q}}$, one is not working with linear theory. However, $\tilde{\gamma}_{\boldsymbol{k}}^{(1)}$ describes first-order response, which justifies the use of formula (203) with Eq. (208).

VII. DISCUSSION

The principal goal of this work was to provide, for the specific example of Hasegawa-Mima dynamics (including the nonadiabatic $k_\parallel = 0$ response), systematic, unambiguous derivations of the contributions to spectral-balance equations from the random interactions of fluctuations of disparate scales. We focused on the long-wavelength growth rate $\gamma_{\boldsymbol{q}}$, but also found expressions for the incoherent noise acting on the long wavelengths as well as the corresponding diffusion and drag terms in the short-wavelength evolution. Energy is properly conserved between the long and short scales. The general results are summarized by Eqs. (66)–(69). Somewhat more explicit formulas for $\gamma_{\boldsymbol{q}}$ are given by Eqs. (22).

Several (related) systematic procedures were employed. In the first (Sec. III), we assumed the validity of the renormalized Markovian closure formulas (15) and (16) and expanded them to lowest nontrivial order in $\epsilon \doteq q/k$. In the second (Sec. IV B), we explicitly performed the functional variations that lead to the Markovian results. In the third (Sec. V), we derived a statistical Poynting theorem and used the functional apparatus to verify that this led to equivalent results.

Heuristically, the statistical Poynting theorem shows that $\gamma_{\boldsymbol{q}}$ is the negative of the variation of the time rate of change of the short-wavelength energy $\mathcal{E}$ with respect to the long-wavelength energy; this is a simple consequence of energy conservation. More formally, $\gamma_{\boldsymbol{q}}$ is proportional to the *second* functional variation of $\partial_T \mathcal{E}$ with respect to the long-wavelength potential $\langle \varphi_{\boldsymbol{q}} \rangle$. That $\gamma_{\boldsymbol{q}}$ is a second-order functional Taylor coefficient shows that intrinsically nonlinear random effects are involved; this is consistent with the very definition of $\gamma_{\boldsymbol{q}}$.

Previous authors have attempted to invoke the properties of linear normal modes and susceptibilities in interpreting the structure and content of formulas like (22). The authors of Ref. [2] appear to have been motivated by the earlier work of Diamond and Kim (DK) [45] on poloidal flow generation due to waves, where an attempt was made to derive a quasilinear Poynting theorem for low-frequency linear normal modes. DK wrote their result in the form

$$\partial_t E_{\boldsymbol{k}} + \nabla \cdot \boldsymbol{S}_{\boldsymbol{k}} + Q_{\boldsymbol{k}} = 0, \tag{217}$$

where $E_{\boldsymbol{k}}$ is the wave energy density, $\boldsymbol{S}_{\boldsymbol{k}} \doteq (\partial \Omega^{\mathrm{lin}} / \partial \boldsymbol{k}) E_{\boldsymbol{k}}$ is the *linear* energy flux, and $Q_{\boldsymbol{k}}$ is the dissipation rate. They subsequently ignored $Q_{\boldsymbol{k}}$, so found that flow generation was driven by the propagation of wave packets at the linear group velocity. It is instructive to compare that approach to the results of the present work. For any ω-independent frequency $\Omega_{\boldsymbol{k}}$ (linear or nonlinear), the rigorous energy conservation law for Hasegawa-Mima dynamics is Eq. (176), which is explicitly

$$\partial_T E_{\boldsymbol{k}} + \nabla \cdot \underbrace{\left(\frac{\partial \Omega_{\boldsymbol{k}}}{\partial \boldsymbol{k}} E_{\boldsymbol{k}} \right)}_{\boldsymbol{S}_{\boldsymbol{k}}} - \underbrace{\frac{\partial}{\partial \boldsymbol{k}} \cdot (\nabla \Omega_{\boldsymbol{k}} E_{\boldsymbol{k}})}_{\text{omitted by DK}} = \underbrace{2 \gamma_{\boldsymbol{k}}^{(1)} E_{\boldsymbol{k}}}_{-Q_{\boldsymbol{k}}}. \tag{218}$$

A partial correspondence between Eqs. (218) and Eq. (217) can be obtained by identifying $2\gamma_{\boldsymbol{k}}^{(1)} E_{\boldsymbol{k}} = -Q_{\boldsymbol{k}}$. The result of DK is missing the $\partial / \partial \boldsymbol{k}$ term [a consequence of inconsistent WKB assumptions about the susceptibility $\chi(\boldsymbol{k}, \omega)$ and potential $\phi(\boldsymbol{k}, \omega)$ in their Eq. (1); they mistakenly ignored $\boldsymbol{X}$ dependence of χ, the first term in the second line of Eq. (145)]. However, we have shown that $\gamma_{\boldsymbol{q}}$ arises specifically from precisely the terms that were neglected by DK. Thus, upon replacing the general frequency Ω by the nonlinear one $\tilde{\Omega}$, one finds that the first-order energy variation obeys

$$\partial_T \hat{\mathcal{E}}_{\boldsymbol{k}} = \nabla \hat{\tilde{\Omega}}_{\boldsymbol{k}} \cdot \frac{\partial E_{\boldsymbol{k}}}{\partial \boldsymbol{k}} + \underline{\nabla \tilde{\Omega}_{\boldsymbol{k}}} \cdot \frac{\partial \hat{\mathcal{E}}_{\boldsymbol{k}}}{\partial \boldsymbol{k}} - \frac{\partial \hat{\tilde{\Omega}}_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \cdot \underline{\nabla E_{\boldsymbol{k}}} - \underline{\frac{\partial \tilde{\Omega}_{\boldsymbol{k}}}{\partial \boldsymbol{k}}} \cdot \nabla \hat{\mathcal{E}}_{\boldsymbol{k}} + 2 \hat{\gamma}_{\boldsymbol{k}}^{(1)} E_{\boldsymbol{k}} + 2 \underline{\tilde{\gamma}^{(1)}} \hat{\mathcal{E}}_{\boldsymbol{k}}. \tag{219}$$

The first, third, and fourth underlined terms vanish at $\tilde{\Omega} = 0$; the second one vanishes because the perturbations are made around a homogenous background. The remaining two terms combine to give

$$\partial_T \hat{\mathcal{E}}_{\boldsymbol{k}} = \nabla \hat{\tilde{\Omega}}_{\boldsymbol{k}} \cdot \left(\frac{1}{k^2} \frac{\partial W_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right); \tag{220}$$

the physics describes the *wave-number* evolution of the weakly inhomogeneous wave packet, not wave-packet propagation. Similarly, the second-order energy variation obeys at $\tilde{\Omega} = 0$

$$\partial_T \hat{\hat{\mathcal{E}}}_{\boldsymbol{k}} + \nabla \cdot \left(\frac{\partial \hat{\tilde{\Omega}}_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \hat{\mathcal{E}}_{\boldsymbol{k}} \right) - \frac{\partial}{\partial \boldsymbol{k}} \cdot (\nabla \hat{\tilde{\Omega}}_{\boldsymbol{k}} \hat{\mathcal{E}}_{\boldsymbol{k}}) = 2 \hat{\gamma}_{\boldsymbol{k}}^{(1)} \mathcal{E}_{\boldsymbol{k}}. \tag{221}$$

Upon summing over $\boldsymbol{k}$ and integrating over $\boldsymbol{X}$, one is led to Eq. (184), in which only the dissipation term survives. The significance of these results is that Eqs. (22) for $\gamma_{\boldsymbol{q}}$, as well as the $\mathsf{D}_{\boldsymbol{k}}$ term to which $\gamma_{\boldsymbol{q}}$ is related, do not vanish even when $\hat{\Omega}_{\boldsymbol{k};\boldsymbol{q}}$ does not depend on $\boldsymbol{k}$ at all (so the group velocity vanishes). Thus calculations or interpretations of $\gamma_{\boldsymbol{q}}$ or $\mathsf{D}_{\boldsymbol{k}}$ that invoke a group velocity, such as the use of Eq. (212), are incorrect.

Let us now comment on an alternative way of writing Eq. (22b) for $\gamma_{\boldsymbol{q}}$. Upon noting that for pure HM the group velocity associated with the nonlinear advection obeys

$$\hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{V}}_{\mathrm{gr},\boldsymbol{k}} = \hat{\boldsymbol{q}} \cdot \frac{\partial}{\partial \boldsymbol{k}} \left(\frac{k_y k^2}{1 + k^2} \right) = \frac{2 k_x k_y}{(1 + k^2)^2}, \tag{222}$$

one finds that Eq. (22b) can be written as

$$\gamma_{\boldsymbol{q}} = -\left(\frac{q^4}{\alpha_{\boldsymbol{q}} + q^2}\right) \sum_{\boldsymbol{k}} k_y \hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{V}}_{\mathrm{gr},\boldsymbol{k}} \theta_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}} \hat{\boldsymbol{q}} \cdot \frac{\partial W_{\boldsymbol{k}}}{\partial \boldsymbol{k}}. \quad (223)$$

Although this form is mathematically correct for the present electrostatic model, we believe that it is misleading in several ways and does not generalize to more complicated situations. First, since we have calculated the group velocity without electric field, one may be tempted to believe that the long-wavelength growth has something to do with linear normal modes; note

$$\hat{\boldsymbol{q}} \cdot \frac{\partial}{\partial \boldsymbol{k}}\left(\frac{k_y k^2}{1+k^2}\right) = -\hat{\boldsymbol{q}} \cdot \frac{\partial}{\partial \boldsymbol{k}}\left(\frac{k_y}{1+k^2}\right), \quad (224)$$

where $k_y/(1+k^2)$ is the linear diamagnetic velocity for unit density scale length. However, we have seen that the true physics involves nonlinear interactions having nothing to do with linear eigenmodes. Second, the presence of a group velocity may suggest that the physics arises from some sort of balance between wave-packet propagation and fluctuation growth, as suggested in Ref. [45]. Although short-wavelength wave packets can propagate in response to a long-wavelength modulation, the analysis in Sec. VI A shows that for $\gamma_{\boldsymbol{q}}$ the effect vanishes on the average; the residual energy loss from the large scales arises from *wave-number* evolution, as described mathematically by Eq. (184) and explained in physical detail by Kraichnan [5]. Finally, if for pure HM dynamics the group velocity should be calculated from the unit advection frequency $k_y k^2 k^2/(1+k^2)$, then for generalized HM dynamics it should be calculated from the unit frequency k_y [cf. Eq. (172)]; however, $\hat{\boldsymbol{q}} \cdot \partial_{\boldsymbol{k}}(k_y) = 0$, so clearly formula (223) does not hold in general.

Thus it is unjustifiable to merely calculate a group velocity (either linear or nonlinear) and somehow insert that into formula (223). What matters is the nonlinear advection frequency, which depends on the form of the nonlinear mode-coupling coefficients; note that changes to the linear dispersion relation leave formulas (22) invariant. Consider, for example, electromagnetic modifications to the electrostatic HM system. Electromagnetic versions of the gyrokinetic equation have been discussed by Hahm *et al.* [46] and Krommes and Kim [47], among others. With $\beta \doteq 4\pi n T_e/B^2 \ll 1$, the new finite-$\beta$ effects are field-line bending in the GK equation and an inductive component to the parallel electric field. As discussed in Ref. [47], it is most natural to introduce a covariant description in terms of the two-vector $(\varphi, A_\parallel)^{\mathrm{T}}$. The resulting system describes both finite-β-modified drift waves as well as shear Alfvén waves, but nonlinearly is somewhat complicated, being in matrix form. For a rough estimate, one may derive a scalar model by focusing on the drift-wave branch. Some simplifications occur for $\beta > m_e/m_i$. In that limit, a highly approximate generalization of the HM equation ($T_i \to 0$) is

$$(1-\nabla_\perp^2-\delta)\partial_t\varphi + i(1-\delta)V_*\partial_y\varphi + \boldsymbol{V}_E \cdot \boldsymbol{\nabla}[-\nabla_\perp^2 + (\omega_*/\omega)\delta]\varphi = 0, \quad (225)$$

where $\delta \doteq \beta(\omega/k_\parallel)^2$ and ω is to be evaluated at the (well-known) linear mode frequency

$$\Omega_{\boldsymbol{k}}^{\mathrm{lin}} = \frac{(1-\delta)\omega_*(\boldsymbol{k})}{1+k_\perp^2-\delta}. \quad (226)$$

Finite β thus introduces a frequency downshift, fundamentally a consequence of a reduced $E_\parallel$ due to Lenz's law in the presence of field-line bending. The advection frequency that follows from Eq. (225) is

$$\tilde{\Omega}_{\boldsymbol{k}} \approx \frac{\boldsymbol{k}\cdot\boldsymbol{V}k^2/(1-\delta)}{1+k^2-\delta}. \quad (227)$$

This is larger than the $\delta=0$ electrostatic result studied earlier in the paper, suggesting according to Eq. (22b) that finite β enhances $\gamma_{\boldsymbol{q}}$ (for $\beta \ll 1$). However, the very crude nature of the approximations made in arriving at Eq. (225) leads us to caution that this result is extremely preliminary and that the details of Eq. (227) cannot be trusted. For present purposes, we merely use this estimate to illustrate the inequivalence of formulas (223) and (22b) when Eq. (227) is used.

We return now to the electrostatic results. Our heuristic algorithms employ only the nonlinear advection frequency and the triad interaction time $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$; no linear effects are in evidence (except through the trivial dependence of θ on $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$). As we noted, the denominator of $\mathcal{R}$ [formula (26), the response function used in Ref. [2] and subsequently [31,32]] contains the linear group-velocity term $qv^{\mathrm{lin}}_{\mathrm{gr},\boldsymbol{k}}$, whereas no such term appears in $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$. In the calculations of Ref. [2], that term arose from the heuristic use of a wave kinetic equation that included linear effects but was not derived from first principles. Now it might be asserted that in the approach of Ref. [2] a multiple-scale approach is used in which only the statistics of the short and rapid scales are averaged over while the long-wavelength fluctuations evolve on an intermediate time scale shorter than the autocorrelation time $\tau_{\mathrm{ac},\boldsymbol{q}}$ for the $\boldsymbol{q}$ fluctuations; if so, then one could reasonably expect the explicit appearance of $qv^{\mathrm{lin}}_{\mathrm{gr},\boldsymbol{k}}$. However, in the fully statistical formalism presented here, the spectral-balance equation for the short scales emerges only after averaging over the long-wavelength statistics; that is, $\boldsymbol{V}_{\boldsymbol{q}}$ is a random variable in the WKE. If that averaging is interpreted as a time average, then it must be over times longer than $\tau_{\mathrm{ac},\boldsymbol{q}}$, so one is no longer studying the intermediate time scale. If the averaging is instead interpreted in the more general ensemble sense, the latter objection does not necessarily apply, but averaging in a homogeneous ensemble removes certain terms in the mean square and one is not free to calculate averages from nonsystematically derived WKE's. The equivalence between our heuristic algorithms and the rigorous asymptotic results of Sec. III show (see discussion in Sec. III C) that the linear propagation must be excluded at lowest order. Note that in Ref. [48] the explicit $qv^{\mathrm{lin}}_{\mathrm{gr},\boldsymbol{k}}$ term was used in a central way to drive a parametric instability. While we have not fully analyzed the physics of individual realizations on intermediate time scales, the natural consequences of such an instability would be to drive steady-state turbulence on long time scales. There is no hint of such a parametric drive in the spectral-balance equations we have derived to dominant order in ϵ. Of course, those equations are no longer valid for $\epsilon = O(1)$, but then the entire calculation of $\gamma_{\boldsymbol{q}}$ must be reconsidered.

Under very broad circumstances, $\gamma_{\boldsymbol{q}}$ is positive. (For isotropic situations, it is sufficient, although not necessary, that $\partial Q_k/\partial k < 0$. More generally, $\gamma_{\boldsymbol{q}} > 0$ if $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$ is sensibly independent of $\boldsymbol{k}$; that is the case when the interactions are dominated by large-scale random shear.) If one speaks instead of eddy viscosity ($\nu_{\boldsymbol{q}} = -\gamma_{\boldsymbol{q}}/q^2$), then the small-$q$ eddy viscosity (defined in the statistical sense of Kraichnan [5]) is negative in those situations. Chechkin *et al.* [7] have recently attempted to calculate an eddy viscosity for Rossby- and drift-wave turbulence. Although they find that $\nu_{\boldsymbol{q}}$ is negative in some circumstances, for the case of 2D isotropic Navier-Stokes turbulence they find that it is always positive. That is in disagreement with our results and the earlier one of Kraichnan, which we believe to be correct. In the Appendix we pinpoint the source of the discrepancy.

In summary, our principal results are as follows for generalized Hasegawa-Mima dynamics.

(1) We performed a systematic calculation of the contributions to the nonlinear growth rate $\gamma_{\boldsymbol{q}}$ (where $\boldsymbol{q}$ is not restricted to pure zonal flows) due to interactions with short scales by direct expansion in $q/k \ll 1$ of the general formula (16a) for renormalized damping that emerges from second-order Markovian statistical closure.

(2) We also calculated the incoherent noise on the long wavelengths and, independently, the effects on the short scales of energy-conserving wave-number diffusion (associated with $\gamma_{\boldsymbol{q}}$) and drag (associated with the incoherent noise).

(3) We showed how functional methods can be used to elucidate the physical origins of the various terms.

(4) We derived a statistically averaged Poynting theorem [Eq. (131)] that shows how $\gamma_{\boldsymbol{q}}$ is related to second-order variations of the short-wavelength energy with respect to the long-wavelength potential [Eqs. (137) and (29)].

(5) We derived the proper wave kinetic equations for energy and enstrophy evolution, correcting conceptual and algebraic errors in the classic derivation by Carnevale and Martin [37] and bringing their results into agreement with those of Smolyakov and Diamond [15].

(6) We showed how WKE's based on the nonlinear advection frequency can be used to derive the lowest-order formula for $\gamma_{\boldsymbol{q}}$ by proceeding from either second-order energetics or first-order flux.

(7) We stressed the key role of wave-number evolution and the first-order distension rate $\gamma^{(1)}_{\boldsymbol{k}}$.

(8) We showed that the use of the interaction time $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$ rather than the response function (containing $q v^{\text{lin}}_{\text{gr},\boldsymbol{k}}$) employed in Ref. [2] and elsewhere is essential to provide a consistent description of the lowest-order physics.

(9) We emphasized that $\theta^r_{\boldsymbol{q},\boldsymbol{k},-\boldsymbol{k}}$ must be constructed from renormalized damping rates $\eta^S_{\boldsymbol{k}}$ that respect random Galilean invariance.

(10) We pinpointed the source of the discrepancy in the value of the eddy viscosity between our own work and that of Refs. [8] and [7].

We did not attempt to work out the consequences of our results for model drift-wave spectra—first, because HM dynamics represent a very simplified model; second, because it is not clear that the wave-number ordering $q/k \ll 1$ is the physically relevant one. Recalling the discussion at the beginning of Sec. I C, we stress that $\gamma_{\boldsymbol{q}}$ and $\dot{E}^{\text{noise}}_{\boldsymbol{q}}$ do not describe all of the contributions to the long-wavelength spectral balance. If they did, in the absence of linear dissipation ($\gamma^{\text{lin}}_{\boldsymbol{q}} < 0$) no steady state could exist. Linear dissipation permits a steady state, but one that may strongly depend on the value of that dissipation. However, the effect of interactions of long-wavelength fluctuations with ones of comparable scales remains to be explored [the physics is described by Eqs. (16)] and significant modifications to the calculations may be required for a complete toroidal treatment [49]. Considerable further work must be done before these analytical methods can make quantitative contact with simulation or experimental data.

Noted added in proof. With regard to the discussion of wave kinetic equations in Sec. (VI A), Professor A. Kaufman has, in a private communication, called our attention to the literature on the Weyl calculus, which elegantly formalizes some of the manipulations beginning with Eq. (145). A review was given by S. W. McDonald [Phys. Rep. **158**, 337 (1998)]. The use of the Weyl symbol $A_{\boldsymbol{k}}(\boldsymbol{X})$ [the exact Fourier transform of an operator kernel $A(\boldsymbol{\rho}|\boldsymbol{X})$ with respect to $\boldsymbol{\rho}$] permits concise generalizations through all orders in the inhomogeneity ϵ [for example, the generalization of Eq. (146) is given by McDonald's Eq. (4.29)]. However, the issue of incorporating all first-order corrections is not addressed *per se* by the Weyl calculus. McDonald reviews the traditional derivation of WKE's, in which a dissipative $O(\epsilon)$ correction arises from the anti-Hermitian part of the dielectric operator. In that language, the correction found in Eq. (160) arises instead from the Hermitian part of the dielectric and is nondissipative. It must also be stressed that the usual derivation leading to the linear-theory-based wave action as the natural dependent variable rails for the intrinsically nonlinear problems discuessed here, in which the appropriate conserved quantity such as Z is dictated by properties of the nonlinearity.

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APPENDIX: EDDY VISCOSITY FOR FIXED SHORT-SCALE SPECTRUM

Recently Chechkin *et al.* [7] generalized earlier work of Montgomery and Hatori [8] on the eddy viscosity of the 2D Navier-Stokes equation to the Rossby-wave/Hasegawa-Mima problem. Those authors calculated turbulent damping of a mean field due to interactions with short scales whose statistics are *fixed* (maintained against viscous dissipation by steady external forcing). In the 2D NS limit, they found a *positive* eddy viscosity $\nu_{\boldsymbol{q}}$, whereas in broad circumstances we, in agreement with Kraichnan, find that $\nu_{\boldsymbol{q}}$ is negative. It is interesting to explore the reasons for this disagreement. Chechkin *et al.* did find a negative viscosity for the HM model in some situations, but their formulas are not in general agreement with ours, and the conceptual and algebraic difficulties are best exposed by considering the NS limit. Although it is not necessary, we shall assume isotropy for simplicity.

Let us rewrite Kraichnan's formula, our Eq. (64), as

$$\nu_{\mathrm{NS}}(q|k_{\min}) = \frac{\pi}{4}\int_{k_{\min}}^{\infty} dk\, \theta^{r}_{q,k,k}\frac{dW(k)}{dk}, \tag{A1}$$

where $W(k) \doteq k^4\langle \delta\varphi^2\rangle(k)$ is the vorticity spectrum. One finds that ν_{NS} is unambiguously negative in two general situations. First, as discussed after Eq. (64), if the triad interaction time is independent of k, then $\nu_{\mathrm{NS}}<0$ for any vorticity spectrum $W(k)$ that vanishes at $k=\infty$. Second, for arbitrary $\theta^{r}_{q,k,-k}$, $\nu_{\mathrm{NS}}<0$ if $W(k)$ is monotonically decreasing, as would be the case in the direct enstrophy cascade [17]. Chechkin *et al.*, however, found for the special case of white-noise short-wavelength forcing [their Eq. (B11) with [50] $\gamma_k=\nu k^2$, rewritten in terms of $W(k)$]

$$\nu^{\mathrm{eff}}_{\mathrm{NS}} = \frac{\pi}{4}\int dk\,\frac{W(k)}{\nu k^3}, \tag{A2}$$

where ν is the classical viscosity. Now the approach of Ref. [7] was perturbative and not renormalized, so only zeroth-order Green's functions appeared. To compare Eqs. (A1) and (A2), one should therefore replace $\theta^{r}_{q,k,-k} \to 1/2\nu k^2$. Then

$$\nu_{\mathrm{NS}} \to \frac{\pi}{4}\int_{k_{\min}}^{\infty}\frac{dk}{2\nu k^2}\frac{dW(k)}{dk} \tag{A3a}$$

$$= -\left(\frac{\pi}{4}\right)\left(\frac{W(k)}{2\nu k^2}\right)\Bigg|_{k_{\min}} + \frac{\pi}{4}\int_{k_{\min}}^{\infty} dk\left(\frac{W(k)}{\nu k^3}\right). \tag{A3b}$$

The last term of Eq. (A3b) is identical to Eq. (A2). Thus the result of Chechkin *et al.* (and that of Montgomery and Hatori as well) is missing the surface contribution that arises in transforming Eq. (A1) by integration by parts. Indeed, the value of that term is just the negative of Chechkin's $\nu^{(3)}$. It is interesting that Chechkin *et al.* wrote Eq. (A2) without explicit limits of integration.

We thus inquire into the origins of the surface term in our calculations. Upon referring to the algebra in Sec. III, one finds that the surface term is just the contribution from region D. Indeed, Eq. (64) was put into its final form involving dW/dk after an integration by parts with a surface correction that was canceled by region D; see the discussions of Eqs. (57) and (85)–(87). The methodology of Ref. [7] misses that contribution. This appears to be related to the assumption that the short-wavelength statistics do not respond to long-wavelength modulation. Thus, in formulas like Eq. (A13) of Chechkin *et al.*, only wave vectors $\boldsymbol{k}$ and $\boldsymbol{k}'$ enter ($\boldsymbol{k}$ and $\boldsymbol{p}$ in our notation) and are the negatives of each other for homogeneous short-wavelength statistics. Region D arises from careful consideration of the triad relation $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}$, so that $\boldsymbol{p}\neq-\boldsymbol{k}$ in the presence of the modulation.

Montgomery and Hatori [8] properly stressed that the assumption of fixed short-wavelength spectrum can be true only in an initial-value sense. Calculations made under that assumption may therefore have some merit for assessing transient effects. However, the energy-conserving calculations presented in the present paper are appropriate for assessing steady or quasisteady states of turbulence in which all scales are interacting self-consistently.

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